(b) Illustrate with an example of a group G having a normal subgroup H such that H and G/H are both nilpotent but G is not nilpotent. 3

(c) Let C_n denote any (up to isomorphism) cyclic group of order n. Write two composition series of C_{12} and verify that they are equivalent. 2

7. *(a)* Write , with justification, a solvable series for the dihedral group D_n (with $2n$ elements). 2

(b) Prove that every PID is Noetherian.Give an example to illustrate that a PID need not be Artinian. 2+2

(c) Prove that every finite p-group is nilpotent. 4

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Ex/SC/MATH/PG/CORE/TH/01/2025

Master of Science Examination, 2025

(1^{st} Year, 1^{st} Semester)

Mathematics

Paper : PG/CORE/TH/01

(Abstract Algebra)

Time : 2 *Hours* *Full Marks* : 40

The figures in the margin inidicate full marks.
Unexplained Symbols/Notations have their usual meanings.
Speical credit will be given for precise answer.

Answer *any* ***four*** questions. 10×4=40

1. Give examples, with proper explanation, of

(a) a simple field extension which is not a finite extension. 1

(b) a separable field extenstion which is not a normal extension. 1

(c) a non-separable field extension. 3

(d) an infinite extension field K/F such that K/E is finite where $E(\neq K)$ is an intermediate field of K/F. 2

(e) an algebraic extension field which is not a finite extension. 3

2. *(a)* Suppose K is a finite field extension of a field F. Write (no proof is required) four equivalent conditions for K to be a Galois extension of F. 3

(b) Suppose F is a finite field with characterstic p. Prove that

(i) $F=F^p$.

(ii) any polynomial in x^p over F is reducible.

(iii) every algebraic extension of F is separable.

3+2+2

3. Suppose K is a field with 3^{2025} elements and F is a field with 3^{75} elements.

(a) Prove that K is an extension of F. Determine $(K : F)$.

4

(b) Prove that K is a Galois extension of F. Find the Galois group $Gal\,(K/F)$ of K over F. Verify if this group is nilpotent, solvable. 5

(c) Using the fundamental theorem of Galois theory or otherwise determine the number of strict intermediate of K/F. 1

4. *(a)* Let ω be a complex cube root of unity and $\sqrt[3]{2}$ be the real cube root of 2. Prove that any two of the three fields $\mathbb{Q}\left(\sqrt[3]{2}\right), \mathbb{Q}\left(\sqrt[3]{2}\omega\right), \mathbb{Q}\left(\sqrt[3]{2}\omega^2\right)$ are isomorphic. Is any of them a Galois extension of $\mathbb{Q}$? Justify your answer. 2+2

(b) Suppose F is any field and n is a positive integer such that char F does not divide n. Prove that the n^{th} roots of unity of F form a cylic group of order n. What is meant by a primitive n^{th} root of unity? What is meant by the n^{th} cyclotomic extension of F? If $F=\mathbb{Q}$ and $n=5$ then what is the Galois group of the corresponding cyclotomic extension over F? 3+1+1+1

5. *(a)* Suppose K is a finite field extension of an infinite field F. Prove that K is a simple extenstion of F if and only if there are finitely many intermediate fields. 5

(b) Give an example, with proper explanation, of a finite extension field which has infinitely many intermediate fields. 5

6. *(a)* What is meant by the derived series of a group G? Define the solvability of G in terms of its derived series. Use this to prove that S_3 is not solvable. 2+1+2