3. *(a)* State Borel's Normal Number theorem.

(b) In the context of part *(a)* above, give a graphical proof of Markov's Inequality: If X is a non-negative step function with range in (0, 1] and if $\alpha > 0$, then

$$\text{Prob }\left[X > \alpha\right] \leq \frac{E(x)}{\alpha}$$ 3+7=10

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Ex/SC/MATH/PG/CORE/TH/05/2025

MASTER OF SCIENCE EXAMINATION, 2025

(1st Year, 1st Semester)

MATHEMATICS

PAPER : PG/CORE/TH/05

(Differential Geometry, Probability and Stochastic Process)

Time : 2 *Hours* *Full Marks* : 40

Symbols/Notations have their usual meanings.

Use separate Answer Script for each Part.

PART—I (20 Marks)

Answer *any* ***four*** of the followings :

1. *(a)* Find the Christoffel symbols of first and second kinds corresponding to the metric

$$ds^2 = (dx')^2 + G(x', x^2)(dx^2)^2$$

where G is a smooth function of x', x^2.

(b) If A_{ij} is a skew-symmetric covariant tensor of rank 2, then prove that

$$\left(\delta^i_j\,\delta^k_l + \delta^i_l\,\delta^k_j\right)A_{ik} = 0$$ 4+1

2. State and prove Ricci's theorem for covariant derivative of tensors. 5

3. Find the curvature at any point of the space curve

$$C : x' = a,\ x^2 = t,\ x^3 = bt$$

where the metric is $ds^2 = (dx')^2 + (x')^2(dx^2)^2 + (dx^3)^2$ and a, b are non-zero constants. 5

4. *(a)* Find the differential equations of the geodesic for the metric $ds^2 = (du)^2 + \sin^2 u(dv)^2$ on a surface.

(b) Show that $g_{ij}\frac{\delta\mu^i}{\delta\beta}\frac{\delta\mu^j}{\delta\beta} = k^2 + \tau^2$, where k and τ are the curvature and torsion of a space curve and (μ^i) are components of prinicipal normal. 3+2

5. Calculate the second fundamental form for the surface $r = (u\cos v, u\sin v, cv)$. Also identify the surface. 4+1

6. Find the Gaussian curvature of the surface defined by

$$x' = f_1(u'),\ x^2 = f_2(u'),\ x^3 = u^2,$$

where f_1, f_2 are differentiable functions. State the geometrical significance of the result. 4+1

PART—II (20 Marks)

Attempt ***any* two** questions. Each quesion carries 10 (ten) marks.

1. Let $\{X_n : n = 0,1,2,...\}$ be a Markov Chain with state space $\mathscr{S} = \{1,2,3,4\}$. Let $\mathcal{P}$ be the one-step transition matrix of $\{X_n\}$, where

$$\mathcal{P} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let A be a non-empty subset of $\mathscr{S}$.

(a) Define T_A, the hitting time to A.

(b) Find Prob $[T_3 < +\infty \mid X_0 = 1]$

(c) Find Prob $[T_1 < +\infty \mid X_0 = 1]$

(d) Find Prob $[T_4 < +\infty \mid X_0 = 2]$ 1+3+3+3=10

2. *(a)* Define a Markov Chain.

(b) Find all the Recurrent States of the Markov Chain with transition Matrix $(\mathscr{S} = \{1,2,3,4\})$

$$\mathcal{P} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

3+4=10