Master of Science Examination, 2025

(1^{st} Year, 1^{st} Semester)

MATHEMATICS

Paper : CORE/TH/02

(Real Analysis)

Time : 2 *Hours* | *Full Marks* : 40

Answer Question No.1 and *any* **four** from the rest.

1. *(a)* If a set A has Lebesgue outer measure zero then prove that the set $\{x^2 : x \in A\}$ also has outer measure zero.

(b) Show that every countable set is Lebesgue measurable.

(c) If f and g are Lebesgue measurable functions defined on a measurable set $D \subset \mathbb{R}$, then prove that the set $\{x \in D : f(x) = g(x)\}$ is measurable. 4+1+3

2. *(a)* If $\{E_n\}_n$ is a sequence of pairwise disjoint measurable sets, then prove that for any $D \subset \mathbb{R}$,

$$\mu^*\left(A \cap (\bigcup_n E_n)\right) = \sum_n \mu^*\left(A \cap E_n\right)$$

and hence show that $\bigcup_n E_n$ is also measurable.

(b) If $E_1 \supset E_2 \supset ... \supset E_n \supset ...$ is a sequence of measurable sets for which $\mu\left(E_k\right) < \infty$ for some k, then prove that $\mu\left(\lim_n E_n\right) = \lim_n \mu(E_n)$. 5+3

3. *(a)* Define sum modulo one " $\dot{+}$ " in [0, 1) and show that if $E \subset [0,1)$ is measurable, then $E \dot{+} y$ is also measurable for $y \in [0,1)$ and $\mu(E \dot{+} y) = \mu(E)$.

(b) If a function *f* is continuous almost every where on a measurable set *D*, then prove that *f* is measurable. 5+3

4. *(a)* Define convergence in measure for a sequence of measurable functions and show that if $\{f_n\}_n$ is convergent to both *f* and *g* in measure, then *f* = *g* almost everywhere.

(b) If a sequence of measurable functions $\{f_n\}_n$ converges almost uniformly to *f*, then show that f_n converges to *f* almost everywhere as also in measure. 4+4

5. *(a)* Prove that for a non-negative measurable function *f* defined on a measurable set *E* of finite measure, $\int_E f d\mu = 0$ iff $f = 0$ almost everywhere on *E*.

(b) Prove that *f* is a function of bounded variation on[*a*, *b*] iff *f* can be expressed as the difference of two increasing functions. 4+4

6. State and prove Monotone Convergence theorem. 8

7. *(a)* If $f : [a,b] \to \mathbb{R}$ is non-decreasing, then prove that f' is Lebesgue integrable and $\int_a^b f' d\mu \le f(b) - f(a)$.

(b) Prove that *S* is a σ-ring iff it is a ring and a monotone class. 6+2

8. *(a)* If μ be a measure defined on a ring *R*, then for any $A \in H(R)$, show that

$$\mu^*(A) = \inf\{\overline{\mu}(F) : A \subset F, F \in S(R)\}$$
$$= \inf\{\overline{\mu}(F) : A \subset F, F \in \overline{S}\}$$

(b) If $E \in H(R)$ has σ-finite outer measure, then prove that *E* must have a measurable cover *F* with $\mu^*(E) = \overline{\mu}(F)$. 4+4

★★★