Master of Science Examination, 2025

(1^{st} Year, 1^{st} Semester)

MATHEMATICS

Paper : CORE/TH/03

(Topology)

Time : 2 *Hours* *Full Marks* : 40

Answer *any* **five** questions

1. Consider the following topology on $X = \{a, b, c, d, e\}$ and let

$$\mathcal{T} = \{X, \phi, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,c,e\}\}$$

(i) List the closed subsets of X. 3

(ii) Determine the closure of the sets $\{a\}$, $\{b\}$ and $\{c,e\}$. 3

(iii) Which sets in *(ii)* are dense in X? 2

2. Let $\mathcal{T} = \{\mathbb{R}^2, \phi\} \cup \{G_k : k \in \mathbb{R}\}$ be the class of subsets of the plane $\mathbb{R}^2$ where $G_k = \{(x, y) : x, y \in \mathbb{R}, x > y + k\}$.

(i) Prove that $\mathcal{T}$ is a topology on $\mathbb{R}^2$. 4

(ii) If X is countable, describe the topology determined by $\mathcal{T}$. 4

3. *(i)* Determine the topology $\mathcal{T}$ on the real line $\mathbb{R}$ generated by the class $\mathcal{A}$ of all closed intervals $[a, a+1]$ with length one. 3

(ii) Let $X = \{a,b,c,d,e\}$ and let $\mathcal{A} = \{\{a,b,c\}, \{c,d\}, \{d,e\}\}$. Find the topology on X generated by $\mathcal{A}$. 3

(iii) Prove that continuous image of a compact set is compact. 2

4. *(i)* Show that if $\mathcal{S}$ is a subbase for topologies $\mathcal{T}$ and $\mathcal{T}^*$ on X, then $\mathcal{T} = \mathcal{T}^*$. 3

(ii) Consider the topology $\mathcal{T} = \{X, \phi, \{a\}, \{b,c\}\}$ on $X = \{a,b,c\}$ and the topology $\mathcal{T}^* = \{Y, \phi, \{u\}\}$ on $Y = \{u,v\}$. 2+3=5

(a) Determine the defining subbase $\mathcal{S}$ of the product topology on $X \times Y$.

(b) Determine the defining base $\mathcal{B}$ of the product topology on $X \times Y$.

5. *(i)* Show that any subspace $(Y, \mathcal{T}_Y)$ of a finite countable space $(X, \mathcal{T})$ is also first countable. 2

(ii) Let $\mathcal{A}$ be any subset of a second countable space X. Then $\mathcal{G}$ is an open cover of $\mathcal{A}$, then $\mathcal{G}$ is reducible to a countable cover. 4

(iii) Show that every path connected space is connected. Is the converse true? Justify your answer. 2

6. *(i)* Let $\mathcal{A} = \{\mathcal{A}_i\}$ be a class of connected subsets of X with a non-empty intersection. Prove that $B = \bigcup_i A_i$ is connected. 3

(ii) Show by a counterexample that a subspace of a separable space need not be separable, i.e. separability is not a hereditary property. 2

(iii) Give example to illustrate that a countable compact set need not be compact or sequentially compact. 3

7. *(i)* Show that the property of being Hausdorff space is topological. 3

(ii) Prove that the following statements are equivalent : 5

(a) X is compact.

(b) For every class $\{F_i\}$ of closed subsets of X, $\bigcap_i F_i = \phi$ implies $\{F_i\}$ contains a finite subclass $\{F_{i_1}, \ldots, F_{i_m}\}$ with $F_{i_1} \cap \ldots \cap F_{i_m} = \phi$.

8. Let X be a topological space.

(i) Give an example of a T_0 space which is not a T_1 space. 3

(ii) Show that the following conditions are equivalent : 5

(a) X is disconnected.

(b) There exists a non-empty proper subset of X which is both open and closed.

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