Master Of Science Examination, 2025

(2^{nd} Year, 1^{st} Semester)

MATHEMATICS

Paper : DSE/02C

(Algebraic and Differential Topology)

Time : *Two Hours* *Full Marks* : 40

*Answer any **five** questions taking at least **two** from each Group.*

Group—A

1. *(a)* Give an example to show that the quotient space of a Hausdorff space may not be Hausdorff.

   (b) Let $\rho \subset X \times X$ be an equivalence relation on a topological space X and let $q : X \to Q = X/\rho$ be the quotient map. If Q is Hausdorff, then show that ρ is a closed subspace of $X \times X$. The converse holds if q is an open map.

   4+4

2. *(a)* When are two paths f and f' from an initial point x_0 to terminal point x_1 in a topological space said to be path homotopic? Show that the relation $\simeq_p$ (being path homotopic) is an equivalence relation.

   (b) Give an example of a topological space and three paths f, g and h such that f and g have straight line homotopic which f and h are not. 6+2

3. *(a)* Show that a contractible space must be path connected.

   (b) Prove that $S^n \backslash S^m$ is homotopy equivalent to S^{n-m-1} $(n > m)$. 2+6

4. Let $h : S^1 \to X$ be a continuous map. Show that the following conditions are equivalent :

   (a) h is null homotopic;

   (b) h extends to a continuous map $k : B^2 \to X$;

   (c) h_* is the trivial homomorphism of fundamental groups. 8

Group—B

5. *(a)* Let $f : R^n \to R^m$ be differentiable at $\tilde{a} \in R^n$ and $g : R^m \to R^p$ be differentiable at $\tilde{b} = \text{f}(\tilde{a}) \in R^m$. Then prove that $h = g \circ f : R^n \to R^p$ is differentiable at $\tilde{a}$.

   (b) Let $f : R^n \to R$ be such that $| \ f(x) \ | \leq \| \ x \ \|^2 \ \forall$ $x = (x_1, ..., x_n) \in R^n$. Then show that f is differentiable at the origin. 6+2

6. Let $f : S \subset R^n \to R^n$ be continuous and has finite partial derivatives with $J_f(\tilde{x}) \neq 0$ for all $\tilde{x} \in S$ where S is open in R^n. If f is one-one on S, then prove that $f : S \to R^n$ is an open mapping. 8

7. Let $H_a = \{(x, y, z) \in R^3 : x^2 + y^2 + z^2 = a\}$, where $a \geq 0$. Prove that H_a is a smooth manifold for $a > 0$ and is not smooth when $a = 0$. 8

8. What is an immersion? State and prove Local Immersion Theorem. 8

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