Master Of Science Examination, 2025

(2nd Year, 1st Semester)

MATHEMATICS

Paper : DSE-03

(Advanced Rings and Modules – I)

Time : *Two Hours* *Full Marks* : 40

Symbols/Notations have their usual meanings

[Let R be a commutative ring with identity 1]

Answer *any* **four** questions : 10×4=40

1. *(a)* Define exact sequence of R-modules and R-homomorphisms. Show that the sequence $O \longrightarrow M \xrightarrow{f} N \longrightarrow O$ of R-modules and R-homomorphisms is exact if and only if f is an isomorphism. Let $O \to M \to O$ be an exact sequence. Show that $M = 0$.

   (b) Consider the following short exact sequence of R-modules and R-homomorphisms $O \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow O$ with M_1 and M_2 are finitely generated R-modules. Show that M is a finitely generated R-module.

   Is the short exact sequence $O \longrightarrow 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \longrightarrow O$ of $\mathbb{Z}$-modules and $\mathbb{Z}$-homomorphisms, a split exact sequence? Justify.

   (2+2+1)+(3+2)=10

2. *(a)* Does there exist a module which is both projective and injective? Justify.

(b) Does there exist a module which is neither projective nor injective? Justify.

(c) With proper justification —

(i) give an example of a projective module which is not an injective module;

(ii) give an example of an injective module which is not a projective module.

(d) Is the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ injective? Justify.

2+2+(2+2)+2=10

3. *(a)* Define tensor product of modules. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

(b) Define character module. State a necessary and sufficient condition in terms of character module so that an *R*-module to be a flat module.

(c) Is the $\mathbb{Z}$-module $\mathbb{Z}_n$ a flat module? Justify your answer.

(2+3)+(2+1)+2=10

4. *(a)* Define co-maximal ideal of *R*. If *A* and *B* are co-maximal ideals of *R*, then show that $AB = A \cap B$. Is this result true, if *A* and *B* are distinct maximal ideals of *R*? Justify.

(b) Define local ring. Let *R* be a local ring. Show that —

(i) *R* has no idempotent element except 0 and 1;

(ii) for $a, b \in R$, $a + b = 1$ implies that either a is a unit or b is a unit.

(1+3+1)+(1+2+2)=10

5. *(a)* Define module of fractions. Let *S* be a multiplicatively closed subset of *R* and *M* be an *R*-module. Show that $S^{-1}R \otimes_R M \cong S^{-1}M$.

(b) What do you mean by radical of an ideal *I* of *R*? When is an ideal *I* of *R* said to be radical? Show that an ideal *I* of *R* is radical if and only if *R*/*I* has no non-zero nilpotent elements.

(2+3)+(1+1+3)=10

6. *(a)* Define prime ring. Let *P* be a proper ideal of *R*. Show that *P* is a prime ideal of *R* if and only if *R*/*P* is a prime ring.

(b) Show that —

(i) an ideal $n\mathbb{Z}$ of the ring $\mathbb{Z}$ is semiprime if and only if $n = 0$ or n is a square free integer;

(ii) a finite field admits no non-zero derivation.

(1+4)+(4+1)=10

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