Master Of Science Examination, 2025

(2^{nd} Year, 1^{st} Semester)

MATHEMATICS

Paper : DSE-03 (A16)

(Operator Algebra I)

Time : *Two Hours* *Full Marks* : 40

Notations *: Here, the ground field for all algebras is the complex field* $\mathbb{C}$ *and* $\sigma_{\mathscr{A}}(a)$ *denotes the spectrum of the element a with respect to the Banach algebra/ C^*-algebra* $\mathscr{A}$.

Attempt *any* **four** questions (10 × 4 = 40)

1. *(a)* Let $\mathscr{A}$ be a unital Banach algebra and $G(\mathscr{A}) = \{a \in \mathscr{A} : a \text{ is invertible in } \mathscr{A}\}$. Prove that the inversion $x \to x^{-1}$ is continuous on $G(\mathscr{A})$.

(b) Let K be a compact Hausdorff space and $\mathscr{A} = C(K)$ be the space of all complex valued continuous functions on K. Then clearly $C(K)$ is a Banach space with the sup-norm.

(i) Prove that with respect to the pointwise multiplication of the functions, $\mathscr{A}$ is a commutative unital Banach algebra.

(ii) If $f \in C(K)$, then find $\sigma_{\mathscr{A}}(f)$. [5+5]=10

2. Let $\mathscr{A}$ be a unital commutative Banach algebra, $\mu_{\mathscr{A}}$ be the set of characters on $\mathscr{A}$ and $C(\mu_{\mathscr{A}})$ be the space of all continuous functions on $\mu_{\mathscr{A}}$.

   (a) Prove that the Gelfand transform $\Gamma : \mathscr{A} \to C(\mu_{\mathscr{A}})$ defined by $\Gamma(a)(\phi) = \phi(a), a \in \mathscr{A}, \phi \in \mu_{\mathscr{A}}$ is a contractive Banach algebraic homomorphism from $\mathscr{A}$ to $C(\mu_{\mathscr{A}})$.

   (b) Prove or disprove : The Gelfand transform $\Gamma : \mathscr{A} \to C(\mu_{\mathscr{A}})$ is always an isometry. [5+5]=10

3. *(a)* If I is a maximal ideal of a unital commutative Banach algebra $\mathscr{A}$, then prove that I is closed.

   (b) Let $\mathscr{A}$ and $\mathscr{B}$ be two unital C^*-algebras and $\pi : \mathscr{A} \to \mathscr{B}$ be a unital $*$-homomorphism. Prove that π is contractive.

   (c) Prove that every self-adjoint element a in a unital C^*-algebra $\mathscr{A}$ is a linear combination of two unitary elements. [3+3+4]=10

4. *(a)* Let a be a positive element in a unital C^*-algebra $\mathscr{A}$. Prove that there exists a unique positive element b in $\mathscr{A}$ such that $b^2 = a$.

   (b) Let a be a normal element in a unital C^*-algebra $\mathscr{A}$ and $f : \sigma_{\mathscr{A}}(a) \to \mathbb{C}$ be any continuous function. Prove that $\sigma_{\mathscr{A}}(f(a)) = \{f(\lambda) : \lambda \in \sigma_{\mathscr{A}}(a)\}$. [6+4]=10

5. *(a)* Prove that for a unital C^*-algebra $\mathscr{A}$ there exists a pair (H, π), where H is a complex Hilbert space and $\pi : \mathscr{A} \to B(H)$ is an injective $*$-homomorphism.

   (b) Let $(H, <\cdot,\cdot>)$ be a complex Hilbert space and $B(H)$ be the C^*-algebra of all bounded linear operators on H. If x is a unit vector in H, then prove that $\phi(T) = <Tx, x>$ is a state on $B(H)$.

   (c) Suppose a is a normal element in a unital C^*-algebra $\mathscr{A}$ and $\sigma_{\mathscr{A}}(a)$ is contained in the imaginary axis of the complex plane. Show that $a^* = -a$. [6+2+2]=10

6. *(a)* Let $\mathscr{A}$ be a unital C^*-algebra and $a \in \mathscr{A}$. Prove that there exists a state ϕ on $\mathscr{A}$ such that $\phi(a^* a) = \|a\|^2$.

   (b) Let a be a self-adjoint element in a unital C^*-algebra $\mathscr{A}$. Prove that there exist two positive elements a^+ and a^- in $\mathscr{A}$ such that $a = a^+ - a^-$. [7+3]=10

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