(b) Use Gauss's test to examine the convergence of the following series :

$$1+\frac{2^2}{3^2}+\frac{2^2}{3^2}\frac{4^2}{5^2}+\frac{2^2}{3^2}\frac{4^2}{5^2}\frac{6^2}{7^2}+\ldots$$

(c) Show that the series $\sum_{n=1}^{\infty}\frac{x^n}{n!}$ is convergent for all real values of x. 2+1+2=5

[CO3]

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Ex/SC/MATH/UG/MAJOR/TH/11/101/2025

BACHELOR OF SCIENCE EXAMINATION, 2025

(1st Year, 1st Semester)

MATHEMATICS

PAPER : MAJOR/TH/11/01

(Real Analysis)

Time : *Two Hours* *Full Marks* : 40

Symbols and Notations have their usual meanings.

Use separate Answer Scripts for each Group.

GROUP—A (20 Marks)

1. Answer *any* **one** question : 5×1=5

(a) (i) If x, y are real numbers with $x<y$, then prove that there exists a rational number r such that $x<r<y$. [3]

(ii) Let
$$A=\{x\in\mathbb{R}: x^2-9x+18\le 0,\ x^2-7x+12\le 0\}$$
Find sup A and inf A. [2]

(b) *(i)* Prove that the set $\mathbb{Q}$ of all rational numbers is denumerable. [3]

(ii) Prove or disprove : If X is an infinite countable bounded subset of $\mathbb{R}$, then the closure of X is countable. [2] **[CO1]**

2. Answer *any* **three** questions : 5×3=15

(a) *(i)* Prove that there does not exist any non-empty proper subset of $\mathbb{R}$ that is both open and closed in $\mathbb{R}$. [4]

(ii) Prove that the interval $I = (0, 2]$ is not an open set in $\mathbb{R}$. [1]

(b) *(i)* Prove that a set $F \subseteq \mathbb{R}$ is closed if and only if the limit of every convergent sequence in F belongs to F. [3]

(ii) Let $A = \left\{\frac{1}{m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\right\}$. If $k \in \mathbb{N}$, show that $\frac{1}{k}$ is a limit point of A. [2]

(b) Find the greatest and least subsequential limit of $\{(-1)^n\}$ and hence show that the sequence $\{(-1)^n\}$ is not convergent. 3+(1+1)=5 **[CO3]**

5. *(a)* Let $\sum_{n=1}^{\infty} u_n$ be a series of positive real numbers such that $\lim_{n\to\infty} (u_n)^{1/n} = l$. Prove that $\sum_{n=1}^{\infty} u_n$ is convergent or divergent according to whether $l < 1$ or $l > 1$.

(b) State general form of Cauchy's root test for a series of positive real numbers.

(c) Use *(b)* to test the convergence of the series $\sum_{n=1}^{\infty} u_n$, where $u_{2n-1} = \left(\frac{1}{2}\right)^{2n-1}$ and $u_{2n} = \left(\frac{1}{5}\right)^{2n}$ for all $n \in \mathbb{N}$.

3+(1+1)=5 **[CO3]**

6. *(a)* Let $\sum_{n=1}^{\infty} u_n$ be a series of real numbers. If $\sum_{n=1}^{\infty} u_n$ is absolutely convergent and $\{v_n\}$ is a bounded sequence of real numbers, then prove that $\sum_{n=1}^{\infty} u_n v_n$ is convergent.

(c) (i) If $K \subseteq \mathbb{R}$ is compact, then prove that K has a maximum and minimum. [3]

(ii) Prove or disprove : Let $A_i = \left(\frac{1}{i}, 2\right)$. Then $C = \{A_i : i \in \mathbb{N}\}$ is an open cover of (0, 1]. [2]

(d) (i) Let $\{[a_n, b_n] : n \in \mathbb{N}\}$ be a family of nested closed and bounded intervals. Prove that $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is non-empty. [3]

(ii) Prove or disprove : The union of an arbitrary collection of compact sets in $\mathbb{R}$ is compact. [2]

[CO2]

GROUP—B (20 Marks)

Answer *any* **four** questions : 5×4=20

1. (a) Let $\{x_n\}$ and $\{y_n\}$ be two sequences such that $\lim_{n\to\infty} x_n = l$ and $\lim_{n\to\infty} y_n = m$. Prove that $\lim_{n\to\infty} Cx_n = Cl$ and $\lim_{n\to\infty} (x_n \pm y_n) = l \pm m$ for any scalar C.

(b) Use (a) to prove that $\lim_{n\to\infty} (x_n)^2 = l^2$.

(c) Use (a) and (b) to prove that $\lim_{n\to\infty} x_n y_n = lm$.

3+1+1=5

[CO3]

2. (a) If $|a| < 1$, then prove that $\lim_{n\to\infty} na^n = 0$.

(b) Let $\{x_n\}$ be a monotonically increasing sequences. If $\{x_n\}$ is bounded above, then prove that it converges to its least upper bound.

(c) Prove that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ 2+2+1=5

[CO3]

3. (a) Let $x_1 = \sqrt{7}$ and $x_{n+1} = \sqrt{7 + x_n}$. Prove that $\{x_n\}$ converges to the positive root of the equation $x^2 - x - 7 = 0$.

(b) For $a > b > c > 0$, find $\lim_{n\to\infty} (a^n + b^n + c^n)^{1/n}$. 3+2=5

[CO3]

4. (a) Prove that a point α is a subsequential limit of $\{x_n\}$ if and only if $x_n \in N(\alpha; \delta)$ for infinitely many values of n and for all $\delta > 0$, where $N(\alpha; \delta)$ is a δ-neighbourhood of α.