8. *(a)* Define *similar* matrices. Prove that similar matrices have same eigen values. 1+1

(b) Let A be an $n \times n$ real matrix such that $I_n + A$ is invertible, where I_n is the $n \times n$ identity matrix. Define $A_* = (I_n - A)(I_n + A)^{-1}$. Show that $(I_n + A_*)$ is invertible and $(A_*)_* = A$. 3

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Bachelor of Science Examination, 2025

(1st Year, 1st Semester)

MATHEMATICS

Paper : MAJOR/TH/11/02

(Geometry and Linear Algebra)

Time : *Two Hours* *Full Marks* : 40

Symbols and Notations have their usual meanings.

Use separate Answer-Scripts for each Group.

GROUP—A

(20 Marks)

Answer **Q 1**, **Q 5** and **Q 6** and *any* **two** from **Q 2**, **Q 3** and **Q** 4 : 4×5=20

1. If the point lies on the ellipse $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, then prove that its polar with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the ellipse $\frac{a'^2 x^2}{a^4} + \frac{b'^2 y^2}{b^4} = 1$. **[CO1]**

2. Show that the equation of the right circular cylinder whose axis is the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and radius r is given by

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\left[\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{l^2+m^2+n^2}\right]^2=r^2$$

[CO2]

3. A sphere of radius k passes through the origin and meets the axes in A, B, C respectively. Show that the locus of the centroid of the triangle ABC is the sphere $9(x^2+y^2+z^2)=4k^2$. **[CO2]**

4. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the co-ordinate axes at A, B, C. Find the equation of the cone generated by the straight lines drawn from O to meet the circle ABC. **[CO2]**

5. Find the condition, so that the plane $lx+my+nz=p$ touches the central conicoid $ax^2+by^2+cz^2=1$. **[CO3]**

6. If PSP' and QSQ' be two perpendicular focal chords of a conic with focus S, then prove that $\left(\frac{1}{SP.SP'}+\frac{1}{SQ.SQ'}\right)$ is constant. **[CO3]**

(b) Find bases and dimensions of subspaces W_1 and W_2 of $\mathbb{R}^5$ over $\mathbb{R}$, where

$W_1=\{(a_1,a_2,a_3,a_4,a_5)\in\mathbb{R}^5 \mid a_1-a_3-a_4=0\}$ and

$W_2=\{(a_1,a_2,a_3,a_4,a_5)\in\mathbb{R}^5 \mid a_2=a_3=a_4, a_1+a_5=0\}$ 2+2

6. *(a)* Let W_1 and W_2 be two subspaces of a vector space V over F. Find the smallest subspace W of V which contains both W_1 and W_2. 2

(b) Define *rank* and *nullity* of a linear transformation. Let $T:\mathbb{R}^n\to\mathbb{R}^m$ be a linear transformation. Show that $\text{rank}(T)+\text{nullity}(T)=n$, $(m,n\in\mathbb{N})$. 3

[CO4] Answer *any* **one** question :

7. Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be the linear transformation defined by $T(x,y,z)=(4x+z, 2x+3y+2z, x+4z)$.

(a) Find the matrix representation A of T with respect to the standard ordered basis of $\mathbb{R}^3$. 2

(b) Determine whether A is diagonalizable. 3

GROUP—B

(20 Marks)

[CO1] Answer *any* **one** question :

1. *(a)* Define a *vector space* over a field. Let $V = \mathbb{R} \times \mathbb{R}$. For $(a,b), (c,d) \in V$ and $r \in \mathbb{R}$, define $(a,b)+(c,d) = (a+d, b+c)$ and $r(a,b) = (ra, rb)$.

Is V a vector space over $\mathbb{R}$ under the above addition and scalar multiplication? Justify your answer. 1+2

(b) Determine whether the first vector can be expressed as a linear combination of the other two in the list $\{(2,-1,1), (1,2,-3), (1,-3,2)\}$ of vectors in $\mathbb{R}^3$ over $\mathbb{R}$. 2

2. Let V be vector space over a field F.

(a) Define the *null vector* θ and show that it is unique. 2

(b) If $\alpha \in F$ and $v \in V$ be such that $\alpha v = \theta$, then show that $\alpha = 0$ or $v = \theta$ (or both). 3

[CO2] Answer *any* **one** question :

3. *(a)* Define *linearly dependent* and *linearly independent* set of vectors in a vector space over $\mathbb{R}$. Let

$$S = \{(1,3,-4,2), (2,2,-4,0), (1,-3,2,-4), (-1,0,1,0)\} \subseteq \mathbb{R}^4$$

Determine whether S is linearly dependent or independent set of vectors in $\mathbb{R}^4$ over $\mathbb{R}$. 1+2

(b) Let

$$S = \{(2,-3,1), (1,4,-2), (-8,12,-4), (1,37,-17), (-3,-5,8)\} \subseteq \mathbb{R}^3$$

Find a linearly independent subset B of S such that span(B) = span(S). 2

4. *(a)* Is there a finite vector space over an infinite field? 1

(b) Suppose a vector space V over a field F can be generated by a finite set of vectors in V. Then show that a maximal linearly independent set in V is a minimal set of generators of V and vice-versa. 4

[CO3] Answer *any* **one** question :

5. *(a)* Let $W = \{(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \mid a_1 + a_2 + \cdots + a_n = c\}$ for some $c \in \mathbb{R}$ $(n \in \mathbb{N})$. Find the values of c such that W is a subspace of $\mathbb{R}^n$ over $\mathbb{R}$. 1