B. Sc. (Mathematics) Examination, 2025 (OLD)

(1^{st} Year, 1^{st} Semester)

MATHEMATICS (HONOURS)

Paper : CORE/TH/02

(Group Theory – I)

Time : *Two Hours* *Full Marks* : 40

All questions carry equal marks.

Answer *any* **four** questions : 10×4=40

1. *(a)* Define a *group*. Verify whether the following set G of real matrices forms a group under usual matrix multiplication :

$$G = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : A^T A = I_2 \right\}$$

where I_2 is the identity matrix of order 2.

(b) Show that any even permutation in $S_n (n \geq 3)$ is a product of 3-cycles.

2. *(a)* Define the *order* $o(a)$ of an element *a* of a group. Let *G* be a group and $a, b \in G$ be such that $b \neq e, a^3 = e$ and $aba^{-1} = b^2$. Show that $o(b) = 7$.

(b) Prove that every non-commutative group has a proper cyclic subgroup containing more than one element.

3. *(a)* Prove that *every* subgroup of cyclic group is cyclic.

(b) Define the center of a group. Show that the center of a group *G* is a subgroup of *G*.

4. *(a)* Let *G* be a group such that every cyclic subgroup of *G* is a normal subgroup of *G*. Prove that every subgroup of *G* is a normal subgroup of *G*.

(b) Define a *normal subgroup* of a group. Let *H* be a subgroup of a group *G*. If for each *G*, there exists $b \in G$ such that $aH = Hb$, then prove that *H* is a normal subgroup of *G*.

5. *(a)* Define the *kernel* of a homomorphism of groups. Prove that every normal subgroup of a group *G* is a kernel of some homomorphism defined on *G*.

(b) Prove that $7\mathbb{Z} / 56\mathbb{Z} \cong \mathbb{Z}_8$.

6. *(a)* Prove that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ if and only if g.c.d $\{m, n\} = 1$ for any two natural numbers *m*, *n*.

(b) State and prove Lagrange's theorem for finite groups.

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