B. Sc. Mathematics Examination, 2025

(2^{nd} Year, 1^{st} Semester)

MATHEMATICS

Paper : MAJOR/205

(Riemann Integration and Series of Functions)

Time : *Two Hours* *Full Marks* : 40

Use separate Answer-Scripts for each Part.

Symbols and Notations have their usual meanings.

PART—I (20 Marks)

Answer *any* **three** questions from **1** to **5** and **one** question from **6** to **7**.

1. Let a function $f:[a,b]\to\mathbb{R}$ be bounded on $[a, b]$. Then show that f is R-integrable on $[a, b]$ iff for each $\varepsilon>0\ \exists\ a\,\delta(>0)$ such that $U(p,f)-L(p,f)<\varepsilon$ for every partition P of $[a, b]$ with $\|p\|\geq\delta$. 5 **[CO1]**

2. Let $f:[a,b]\to\mathbb{R}$ be R-integrable on $[a, b]$. If there exists a $R\in R^{+}$ such that $f(x)\geq R\ \forall x\in[a,b]$, then prove that $\frac{1}{f}$ is R-integrable on $[a, b]$. 5 **[CO1]**

3. *(a)* Show that $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}$.

(b) If $f:[a,b] \to \mathbb{R}$ and $\phi:[a,b] \to \mathbb{R}$ are both R-integrable on $[a, b]$ and f is monotone decreasing on $[a, b]$, then show that $\exists$ a point ξ in $[a, b]$ such that

$$\int_a^b f(x)\phi(x)dx = f(a)\int_a^{\xi} \phi(x)dx + f(b)\int_{\xi}^b \phi(x)dx \; .$$

3+2=5 **[CO1]**

4. *(a)* If $f:[a,b] \to \mathbb{R}$ is integrable on $[a, b]$, then show that the function F defined by $F(x) = \int_a^x f(t)dt$, $x \in [a,b]$ is continous on $[a, b]$.

(b) Show that $[x]$ is integrable on $[0, 3]$ and evaluate $\int_0^3 [x]dx$. 3+2=5 **[CO1]**

5. *(a)* State Bonnet's form of 2nd M. V. T. in the integral calculus.

(b) The function f defined by

$$f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \le \frac{1}{n} (n = 1,2,3,...) \\ 0, & x = 0 \end{cases} .$$

Show that f is R-integrable on $[0, 1]$. Evaluate $\int_0^1 f dx$.

1+4=5 **[CO1]**

6. Examine the convergence of $\int_0^1 x^{m-1}(1-x)^{n-1} \log x \, dx$.

5 **[CO2]**

(b) Find the radius of convergence of the following power series : 4 **[CO3]**

$$\frac{1^2}{4^2}\frac{x}{3} + \frac{1^2.5^2}{4^2.8^2}\left(\frac{x}{3}\right)^2 + \frac{1^2.5^2.9^2}{4^2.8^2.12^2}\left(\frac{x}{3}\right)^3 + \frac{1^2.5^2.9^2.13^2}{4^2.8^2.12^2.16^2}\left(\frac{x}{3}\right)^4 + ...$$

4. Answer *any* **one** question : 4×1=4

(a) Assuming the power series expansion of $(1+x^2)^{-1}$, obtain the power series expansion of $\tan^{-1}x$ and hence obtain the sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}$.

3+1=4 **[CO4]**

(b) Express $f(x) = x(\pi - x), 0 < x < \pi$ as *(i)* Half-Range Fourier cosine series and *(ii)* Half-Range Fourier sine series by constructing two functions $g(x)$ and $h(x)$ of periodicity 2π, where $g(x)$ is even and $h(x)$ is odd.

2+2=4 **[CO4]**

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7. *(a)* Find $\int_0^1 \frac{dx}{\left(1-x^3\right)^{\frac{1}{3}}}$.

(b) A function f is defined on $[1, \infty)$ by $f(x)=\frac{(-1)^{n-1}}{n}$ for $n \le x < n+1$ $(n=1, 2, 3, ...)$. Examine the convergence of $\int_1^\infty f(x)dx$. 2+3=5 **[CO2]**

PART—II (20 Marks)

1. Answer *any* **one** question : 4×1=4

(a) Using the definition of uniform convergence of sequence of functions, prove that the sequence of functions $\{f_n\}$ defined by $f_n(x)=\tan^{-1} nx, \forall\, x \ge 0, \forall\, n \in \mathbb{N}$ is uniformly convergent on $[a, b]$ for all $b > a > 0$ but the sequence of functions $\{f_n\}$ is not uniformly convergent on $[0, b]$ for any $b > 0$. 3+1=4 **[CO1]**

(b) Show that the sequence of functions $\{f_n\}$ defined by $f_n(x)=x^n, x \in \mathbb{R}$ is point-wise convergent for $x \in (-1,1]$ and in this case find $\lim_{n\to\infty} f_n(x)$. Using the definition of uniform convergence of sequence of functions, prove that the above sequence of functions is uniformly convergent on $[0, b]$ for all $0 < b < 1$ but the sequence of functions $\{f_n\}$ is not uniformly convergent on $[0, 1]$. 1+2+1=4 **[CO1]**

2. Answer *any* **two** questions : 4×2=8

(a) Let $\{f_n\}$ be a sequence of functions defined on $E(\subseteq R)$. If $\{f_n\}$ is point-wise convergent to f on $E(\subseteq R)$ and $M_n = \sup_{x\in E} |f_n(x)-f(x)|$ exists for all $n \in \mathbb{N}$, then $\{f_n\}$ is uniformly convergent to f on E if and only if $M_n \to 0$ as $n \to \infty$. 4 **[CO2]**

(b) Let $\{f_n\}$ be a sequence of functions defined on $E(\subseteq R)$. If $f_n(x) \le M_n \forall\, x \in E$ and $\forall\, n \in N$ and $\sum_{n=1}^{\infty} M_n$ is convergent, then $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $E(\subseteq R)$. 4 **[CO2]**

(c) Let $R > 0$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. Then prove that the radius of convergence of $\sum_{n=1}^{\infty} n a_n x^{n-1}$ = the radius of convergence of $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = R$. 2+2=4 **[CO2]**

3. Answer *any* **one** question : 4×1=4

(a) Show that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on [0, 1], where

$$f_1(x)=\frac{x^2}{1+x},\ f_n(x)=\frac{nx^2}{1+nx}-\frac{(n-1)x^2}{1+(n-1)x} \forall\, n-1 \in \mathbb{N}$$

4 **[CO3]**