5. Suppose $f:[a,b]\to\mathbb{R}$ be differentiable function. Let $a<c<d<b$. If $f'(c)<0$ and $f'(d)>0$, then show that there exists a point $\xi\in(c,d)$ such that $f'(\xi)=0$. 5

6. *(a)* Find the value of a and b so that $\lim\limits_{x\to 0}\dfrac{a\sin 2x-b\sin x}{x^3}=1$

(b) Evaluate $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}$ 2½+2½=5

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B. Sc. Mathematics Examination, 2025 (OLD)

(2nd Year, 1st Semester)

Mathematics

Paper : CORE-5

[Theory of Real Function (Old)]

Time : *Two Hours* *Full Marks* : 40

Use separate Answer Scripts for each part.

Symbols / Notations have their usual meanings.

PART—I (20 Marks)

Answer *any* **four** questions : 5×4=20

1. *(a)* Let $I=(a,b)$ be a bounded open interval and $f:I\to\mathbb{R}$ be a monotone increasing function on I. If f is bounded below on I, then show that $\lim\limits_{x\to a+}f(x)=\inf\limits_{x\in(a,b)}f(x)$. 3

(b) A function $f:\mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x)=0$ for all $x\in\mathbb{Q}$. Prove that $f(x)=0$ for all $x\in\mathbb{R}$. 2

2. *(a)* A function $f:\mathbb{R}\to\mathbb{R}$ is such that $f^{-1}(G)$ is open in $\mathbb{R}$ whenever G is open in $\mathbb{R}$. Prove that f is continuous on $\mathbb{R}$. 3

(b) Let $f:\mathbb{R}\to\mathbb{R}$ be continuous on $\mathbb{R}$. Prove that the set $S=\{x\in\mathbb{R}: f(x)>0\}$ is an open set in $\mathbb{R}$. 2

3. *(a)* Let $[a, b]$ be a closed and bounded interval and $f:[a, b]\to\mathbb{R}$ be continuous and injective on $[a, b]$. Prove that f is strictly monotone on $[a, b]$. 3

(b) A function $f:[0, 1]\to[0, 1]$ is continuous on $[0, 1]$. Prove that there exists a point c in $[0, 1]$ such that $f(c)=c$. 2

4. If $f(x+y)=f(x)+f(y)$ for all $x, y\in\mathbb{R}$ and f is continuous at a point c in $\mathbb{R}$, prove that f is uniformly continuous on $\mathbb{R}$. 5

5. *(a)* Let a function f be continuous on an open bounded interval (a, b) and let $\lim\limits_{x\to a+} f(x)$ and $\lim\limits_{x\to b-} f(x)$ both exist finitely. Prove that f is uniformly continuous on (a, b). 3

(b) Let $f:\mathbb{R}\to\mathbb{R}$ be continuous on $\mathbb{R}$. A point $c\in\mathbb{R}$ is said to be a fixed point of f if $f(c)=c$ holds. Prove that the set of all fixed points of f is a closed set. 2

6. *(a)* Let $D\subseteq\mathbb{R}$ be a compact set and a function $f:D\to\mathbb{R}$ be continuous on D. Prove that $f(D)$ is a compact set in $\mathbb{R}$. 3

(b) Find the points of discontinuity of the function $f(x)=[x]+[-x], x\in\mathbb{R}$, where $[x]$ denotes the greatest integer not greater than x. 2

PART—II (20 Marks)

Answer *any* **four** questions : 5×4=20

1. A function f is defined in $(-1, 1)$ by

$$f(x)=\begin{cases} x^p \sin\left(\dfrac{1}{x^q}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$$

Determine the conditions of p and q when f' is continuous and discontinuous at $x = 0$. 5

2. *(a)* Let a function $f(x)$ be defined in $[0, 2]$ by $f(x)=x-[x]$. Does there exist a function $\varphi(x)$ such that $\varphi'(x)=f(x)$?

(b) If $\varphi(x)$ is a polynomial in x and λ is a real number, then prove that there exists a root of $\varphi'(x)+\lambda\varphi(x)=0$, between any pair of roots of $\varphi(x)=0$. 2+3=5

3. If $\varphi(x)=f(x)+f(1-x)$, $x\in[0,1]$ and $f''(x)<0$, for all $x\in[0,1]$, show that φ is increasing on $\left[0,\frac{1}{2}\right]$ and decreasing on $\left[\frac{1}{2},1\right]$. 5

4. If $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_{n-1}}{2}+a_n=0$, then show that the equation $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=0$ has at least one real root between 0 and 1. 5