(b) Suppose V is a vector space over a field F and $T: V \to V$ is a linear operator.

(i) Prove that eigenvectors corresponding to two distinct eigenvalues of T are linearly independent.

(ii) Suppose V is finite dimensional and $B = \{v_{1,} v_2 \cdots, v_n\}$ is an ordered basis of V such that v_i is an eigenvector of T corresponding to an eigenvalue λ_i, for all $i = 1, 2, \cdots, n$. Find the matrix of T with respect to the basis B. 2+2

(c) Suppose the matrix of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the standard ordered basis of $\mathbb{R}^3$ is $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}$. Find $T(x, y, z)$ for any $(x, y, z) \in \mathbb{R}^3$ and find the matrix B(say) of T with respect to the ordered basis $\{(1, 1, 0), (0, 0, 1) (0, 1, 1)\}$. 2+2

★★★

Ex/SC/MATH/UG/CORE/TH/06/2025(OLD)

B. Sc. Mathematics Examination, 2025 (OLD)

(2nd Year, 1st Semester)

(Ring Theory and Linear Algebra–I)

Paper : CORE-6

Time : 2 *Hours* *Full Marks* : 40

Use separate answer scripts for each part.

Symbols/Notations have their usual meanings.

PART—I (20 Marks)

Answer *any* **four** questions : 5×4=20

1. Let R be a ring with identity 1 and $a, b \in R$.

(i) If $ab + ba = 1$ and $a^3 = a$, then show that $a^2 = 1$.

(ii) If $ab = a$ and $ba = b$, then show that a and b are idempotents in R. 3+2

2. Let R be a ring with identity 1. When is R said to be commutative? Show that R is a commutative ring if and only if $(ab)^2 = a^2b^2$ for all $a, b \in R$. 1+4

3. Define Boolean ring. Show that —

   (i) the characteristic of a Boolean ring is 2;

   (ii) every Boolean ring is commutative. 1+2+2

4. Define integral domain and field. Show that every finite integral domain is a field. 2+3

5. Let R be commutative ring with identity. Define prime ideal and maximal ideal of R. Show that every maximal ideal of R is a prime ideal of R. 2+3

6. State first isomorphism theorem of ring. By using this theorem, show that any epimorphism from the ring $\mathbb{Z}$ onto itself is an isomorphism. 2+3

PART—II (20 Marks)

(Linear Algebra)

1. Answer *any* **two** from the following questions : 4×2=8

   (a) State and prove the rank nullity theorem for a linear transformation. 4

   (b) *(i)* Define the column rank and the row rank of an $m \times n$ matrix A over a field F.

   (ii) Prove that there cannot exist any onto linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^7$. Generalize this result. 1+3

   (c) *(i)* Does there exist a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^4$ such that $T(1,1,0)=(1,2,3,-1)$, $T(0,1,1)=(0,1,3,-2)$? If the answer is yes, is it unique? Answer with precise reason.

   (ii) Find the change of basis matrix P(say) for the change from the basis $B_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$ to $B_2 = \{(1,1,0),\ (0,0,1),\ (0,1,1)\}$ of the vector space $\mathbb{R}^3$. 3+1

2. Answer *any* **one** from the following questions : 4×1=4

   (a) Let the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x,y,z)=(x+y,y,z)$. Find the eigenvalues of T and their geometric and algebraic multiplicities. 4

   (b) Determine the subspaces of the vector space $\mathbb{R}^2$. 4

3. Answer *any* **two** from the following questions : 4×2=8

   (a) Let V be a vector space over a field F and $T:V \to V$ be a linear operator on V. Prove that $\lambda \in F$ is an eigenvalue of T if and only if $\text{Null}(T-\lambda I) \geq 1$. Hence prove that if $\dim(V)$ is finite and rank $(T) < \dim(V)$, then 0 is an eigenvalue of T. 3+1