3. (a) Give an example of a function whose singularity is a branch point.

(b) Suppose $\operatorname{Im}\{f'(z)\} = 6x(2y-1)$ and $f(0) = 3-2i$, $f(1) = 6-5i$. Find $f(1+i)$. 1+4=5

4. Let $w = f(z) = (z^2+25)^{1/2}$.

(a) Show that $z = \pm 5i$ are branch points of $f(z)$.

(b) Show that a complete circuit around both branch points produces no change in the branches of $f(z)$. 3+2=5

5. (a) Investigate the region of absolute convergence of $\sum_{n=1}^{\infty} \frac{n(-1)^n (z-i)^n}{4^n (n^2+1)^{5/2}}$.

(b) Find the region of convergence of $\sum_{n=1}^{\infty} \frac{e^{2\pi i n z}}{(n+1)^{3/2}}$.

3+2=5

6. (a) Suppose $f(z)$ is analytic inside and on a circle C of radius r and center at $z = a$. Then

$$|f'''(a)| \le \frac{6M}{r^3}$$

where M is a constant such that $|f(z)| < M$ on C.

(b) Evaluate $\oint_C \frac{e^{\sin z}}{(z+1)^4}\, dz$, where C is the circle $|z| = \sqrt{3}$. 5

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B. Sc. Mathematics Examination, 2025

(3rd Year, 1st Semester)

Paper Code : Core-12

(Metric Space and Complex Analysis)

Time : 2 Hours *Full Marks* : 40

GROUP—A

(Metric Space)

Answer *any* **four** questions : 5×4=20

[CO1]

1. (a) For any three points x, y, z in a metric space (X,d), prove that $|d(x,z) - d(y,z)| \le d(x,y)$.

(b) If (X,d) is a metric space, then check whether $(X,\bar{d})$ is a metric space or not, where $\bar{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}$.

2+3=5

[CO4]

2. (a) Let X be the set of polynomials $P(t)$, $0 \le t \le 1$ with real coefficients, where the metric d is defined by

$$d(P_1,P_2) = \sup_{0\le t\le 1} |P_1(t) - P_2(t)|$$

Is (X,d) a complete metric space? Justify.

[CO1]

(*b*) Define diameter of a set S in a metric space (X, d), with an example. 3+2=5

[CO2]

3. State and prove Cantor's intersection theorem for metric space (X, d). 5

[CO2]

4. (*a*) When is a set E in a metric space said to be of the second category?

[CO4]

(*b*) Prove that a complete metric space (X, d) is a set of the second category? 1+4=5

[CO3]

5. (*a*) Is (l_2, d) compact with usual metric d? Justify.

(*b*) Prove that a compact metric space is complete. Is the converse true? Justify. 2+2+1=5

[CO3]

6. (*a*) (*i*) When is a point in a metric space (X, d) said to be a fixed point of an operator T on X.

(*ii*) State Banach's fixed point theorem on a metric space. Why is it called contraction mapping principle?

[CO4]

(*b*) Let $X = \left(0, \frac{1}{4}\right)$ be a metric space with usual metric of the set of real numbers. If $T : X \to X$ is given by $Tx = x^2$, then find its fixed point in X. 1+2+2=5

GROUP—B

(Complex Analysis)

Answer *any* **four** questions : 5×4=20

1. Describe the stereographic projection of points on the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ to the extended complex plane $\mathbb{C}_\infty$. Show that under this projection, the point $z = x + iy$ corresponds to the point

$$\left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$$

on the sphere. What corresponds on the sphere to _____

(*a*) straight lines in the z-plane,

(*b*) circles in the z-plane with the center as the origin?

1+2+2=5

2. (*a*) Prove that $\psi = \ln\left[(x-1)^2 + (y-2)^2\right]$ is harmonic in every region which does not include the point $(1, 2)$.

(*b*) Find a function ϕ such that $\phi + i\psi$ is analytic.

(*c*) Express $\phi + i\psi$ as a function of z. 1+2+2=5