# Master of Science Examination, 2024

**(2nd Year, 1st Semester)**

**Mathematics**

**Paper : DSE-04-A – 15**

**( Theory of Semigroups – 1 )**

*Time* : 2 *Hours* *Full Marks* : 40

*Symbols / Notations have their usual meanings.*

*Answer **any four** questions.*

1. *(a)* Define a O-group. Prove that a semigroup with O is a O-group if and only if $aS = Sa = S, \forall a \in S \mid \{0\}$.

   *(b)* Show that a semigroup is isomorphic to a rectangular band if and only if it is isomorphic to the direct product of a left zero semigroup and a right zero semigroup. Is a rectangular band a semilattice? Explain. 4+6

2. *(a)* Define a congruence on a semigroup. Let $S$ and $T$ be two semigroups and $\phi : S \to T$ be a homomorphism. Then show that $\ker\phi = \phi \circ \phi^{-1}$ is a congruence on $S$ and there is a monomorphism $\alpha : S/_{\ker\phi} \to T$ such that $\mathrm{im}\alpha = \mathrm{im}\phi$.

*(b)* Define orthodox semigroup. If $S$ is an orthodox semigroup, $e$ is an idempotent in $S$ and $a \in S$, then show that for every inverse $a'$ of $a$, the elements $a'ea$ and $aea'$ are both idempotent. 6+4

**3.** *(a)* Let $S$ be a semigroup. Define $E(S)$. Give two examples of semigroups $S$ and $T$ such that $E(S) = \phi$ and $E(T) = T$.

*(b)* Define partial mapping $P_x$ on a non-empty set $X$.

If $\phi$ is a partial map, then $\phi^{-1}$ is also a partial map. Is the statement true? Illustrate with an example. If $\phi, \psi \in P_x$, then show that $\mathrm{dom}(\phi \circ \psi) = [\mathrm{im}\phi \cap \mathrm{dom}\psi]\phi^{-1}$.

3+7

**4.** *(a)* Define the Green's equivalence relations $D$ and $J$ on a semigroup. Prove that in a periodic semigroup, $D = J$.

*(b)* Let $a$ be an element of a regular $D$-class $D$ in a semigroup $S$. If $b \in D$ is such that $R_a \cap L_b$ and $L_a \cap R_b$ contains idempotent $e$ and $f$ respectively, then show that $H_b$ contains an inverse $a^*$ such that $aa^* = e$ and $a^*a = f$.

5+5

**5.** *(a)* Let $H$ be an $H$-class of a semigroup $S$. Then prove that either $H^2 \cap H = \phi$ or $H$ is a subgroup of $S$.

*(b)* Let $S$ be a semigroup. Then prove that the following statements are equivalent :

*(i)* $S$ is completely regular,

*(ii)* every element of $S$ lies in a subgroup of $S$;

*(iii)* every $H$-class in $S$ is a group.

(You may use the above result Q5*(a)*) 4+6





**6.** Define a finite monogenic semigroup $S$. Define also index and period of $S$. Show that $S$ has a cyclic subgroup. Describe all the elements of this group. 1+2+4+3

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