Master of Science Examination, 2024

(2nd Year, 1st Semester)

Mathematics

Paper : DSE - 02C

[Algebraic and Differential Topology]

Time : *Two hours* *Full Marks* : 40

*Answer **any five** questions taking at least **two** from each Group*

GROUP—A

1. *(a)* Let X and Y be topological spaces and let $p : X \to Y$ be a surjective continuous map. Then prove that following are equivalent :

   (i) (Y, p) is a quotient space for some equivalence relation ~ on X.

   (ii) $U \subset Y$ is open in Y iff $p^{-1}(U)$ is open in X.

   (iii) $Z \subset Y$ is closed in Y iff $p^{-1}(Z)$ is closed in X.

   (b) Explain the construction of cylinder from unit square briefly. 7+1

2. *(a)* If f is a path in X from x_0 to x_1 and if g is a path in X from x_1 to x_2 define $[f]*[g]$. Given $x \in X$ and where e_x denotes the constant path x, prove that

$$[f]*[e_{x_1}]=[f] \text{ and } [e_{x_0}]*[f]=[f]$$

(b) For three paths f, g, h, if $[f]*([g]*[h])$ is defined, then prove that

$$[f]*([g]*[h])=([f]*[g])*[h].$$ 8

3. *(a)* Find $\pi_1[B^n, 1]$.

(b) Show that $p:\mathbb{R} \to S^1$ given by

$p(x)=(\cos 2\pi x, \sin 2\pi x)$ is a covering map. 2+6

4. *(a)* Show that there is no retraction of B^2 onto S^1.

(b) Use Brouwer fixed point theorem to prove that any 3 by 3 matrix A of positive real numbers has a positive real eigenvalue. 8

GROUP—B

5. *(a)* State and prove Mean Value Theorem for $\mathbb{R}^n$.

(b) Let S be an open connected subset of $\mathbb{R}^n$ and let $f:S \to \mathbb{R}^m$ be differentiable at each point of S. If $f'(\tilde{C})=0\ \forall \tilde{C} \in S$, then show that f is constant on S.

6+2

6. Let $f:\mathbb{R}^n \to \mathbb{R}^n$ has continuous partial derivatives $D_j f_i$ on an open set $S \subset \mathbb{R}^n$ and $J_f(\tilde{a}) \neq 0$ for some point $\tilde{a} \in S$. Then show that $\exists$ an open ball $B(\tilde{a}, r)$ on which f is one-one. Give an example to show that we cannot ensure that f is one-one on whole of $\mathbb{R}^n$ even if $J_f(\tilde{x}) \neq 0\ \forall \tilde{x} \in \mathbb{R}^n$. 8

7. *(a)* Show that $X=\{(x,y)\in \mathbb{R}^2 : xy=0\}$ is not a smooth manifold.

(b) Define the tangent space $T_x(X)$ of a k-dimensional smooth manifold X and show that $T_x(X)$ is well defined i.e., it is independent of the choice of parametrization.

3+5

8. State and prove the Inverse Function Theorem for smooth manifolds. 8

★ ★ ★