Master of Science Examination, 2024

(Second Year, First Semester)

Mathematics

Paper : DSE - 03

[Graph Theory I]

Full Marks : 40 *Time* : 2 *Hours*

Answer ***any four*** *questions* 4×10

1. *(a)* Show that a simple graph with more than one vertex cannot have all degrees of vertices distinct.

   (b) What is a *graphical sequence*?

   (c) Let $n \geq 1$ and $d_1, d_2, \ldots, d_n$ be a non-increasing sequence of non-negative integers such that $1 \leq d_1 < n$. Then show that this sequence is graphical if and only if the sequence $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_{1+2}}, d_{d_1+3}, \ldots, d_n$, when put into non-increasing order, is graphical.

   (d) Determine whether the sequence $(5, 5, 4, 3, 3, 2, 2, 2)$ is graphical or not.

   (e) Let G be a k-regular graph where k is odd. Show that G has even number of vertices. 2+1+4+2+1

2. *(a)* Let G be a simple graph with n vertices. Suppose the degree of each vertex is greater than or equal to $\frac{n-1}{2}$. Then show that G is connected.

(b) If a graph G contains exactly two vertices of odd degree, then show that there exists a path between these vertices.

(c) Define *vertex connectivity* $\kappa(G)$ and *edge connectivity* $\kappa'(G)$ of a simple graph G.

(d) Show that $\kappa(G) \leq \kappa'(G)$ for a simple graph G.

3+2+2+3

3. *(a)* Define an *Eulerian circuit.*

(b) Prove that a connected even graph has an Eulerian circuit.

(c) What is the *Hamiltonian closure* of a simple connected graph?

(d) Let G be a simple connected graph with $n > 2$ vertices and $e > \binom{n-1}{2} + 1$ edges. Show that the Hamiltonian closure of G is complete.

(e) Let A be the adjacency matrix of a simple graph G with n vertices. If $A^2 = (a_{ij})_{n \times n}$, then prove that a_{ii} = the degree of the vertex corresponding to the i^{th} row.

1+3+1+3+2

4. *(a)* What is a *perfect matching*?

(b) Show that a regular bipartite graph has a perfect matching.

(c) Show that every component of the symmetric difference of two matchings is a path or even cycle.

(d) Define the *independence number* of a graph G.

(e) Let G be a bipartite graph with no isolated vertices. Then show that the independence number of G is equal to the minimum size of an edge cover in G.

1+2+2+1+4

5. *(a)* Define a *k-critical graph* for a natural number k.

(b) If H is a k-critical graph, then show that $\delta(H) \geq k-1$, when $\delta(H)$ is the minimum degree of vertices of H.

(c) If G is a graph, then show that the chromatic number $\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$.

(d) Find the chromatic polynomial of a cycle of length n.

(e) Show that the chromatic index of a complete graph with n vertices is n for all odd $n > 1$.

1+2+1+3+3

6. *(a)* Prove that chordal graphs are perfect.

(b) Prove that the complement of an interval graph is transitively oriented.

5+5

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