(b) Prove/Disprove $\mathrm{Re}\left(\int_C f(z)\,dz\right) = \int_C \mathrm{Re}(f(z))\,dz$. 4+1

3. Let $\cot(\pi z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and $\csc(\pi z) = \sum_{n=-\infty}^{\infty} b_n z^n$. Define the Laurent series expansion of $\cot(\pi z)$ and $\csc(\pi z)$, on the annulus $1 < |z| < 2$. Evaluate the coefficients a_{-n} and b_{-n} for $n \in \mathbb{N}$ .5

4. Define radius of convergence and region of convergence of a power series $\sum_{n=0}^{\infty} a_n z^n$. Find the region of convergence of the series $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+2)^3 4^{n+1}}$. 1+4

★★★

B. Sc. Mathematics (Hons.) Examination, 2024

(3rd Semester)

Mathematics (Core)

Paper : Core/Math/Th/12

(Metric Space and Complex Analysis)

Time : *Two Hours* *Full Marks* : 40

(20 marks for each Group)

Symbols/Notations have their usual meanings.

Use separate Answer scripts for each Group.

GROUP—A

(Metric Space)

(20 Marks)

Answer ***any four*** questions

All questions carry equal marks

1. (a) Consider the metric space $(\mathbb{R}, d)$ with usual metric $d(x, y) = |x - y|$. Again consider the discrete d_1 on $\mathbb{R}$. Give an example of an open set in $(\mathbb{R}, d_1)$ that is not open in $(\mathbb{R}, d)$. Does there exist an open set in $(\mathbb{R}, d)$ which is not open in $(\mathbb{R}, d_1)$? Justify.

(*b*) Give an example to show that arbitrary intersection of open sets is not open in a metric space. 3+2

2. (*a*) When is a metric space said to be complete?

(*b*) Show that the space of all real valued continuous functions defined on [0, 1] is not a complete metric space with respect to the metric *d*, defined as

$$d(f, g) = \int_0^1 | f(x) - g(x) | \, dx \qquad 1+4$$

3. (*a*) Let $f:[0,1] \to [0,1]$ and $g:[0,1] \to \mathbb{R}$ be two continuous functions such that $g(0) = 0, g(1) = 1$. Then show that $f(x) = g(x)$ for some $x \in [0,1]$.

(*b*) Give an example of a set *S* in $(\mathbb{Q}, d)$, where $d(x, y) = | x - y |$, such that *S* is closed and bounded but not compact. 3+2

4. (*a*) Show that continuous image of a compact metric space is compact.

(*b*) Let (X, d) be a metric space and *A* be a compact subset of *X*. Let $x_0 \notin A$ and $\text{dist}(x_0, A) = \inf\{d(x_0, a) : a \in A\}$. Show that there exists an element $a_0 \in A$ such that $\text{dis}(x_0, A) = d(x_0, a_0)$. 2+3

5. (*a*) Define connected metric space.

(*b*) Let *S* be a connected subset of the set of rationals with usual metric. Then what can be said about the cardinality of *S*?

(*c*) If *A* is a connected subset of a metric space *X* and $A \subset B \subset \overline{A}$, then show that *B* is connected. 1+1+3

6. Let *E* be a non-compact subset in $\mathbb{R}$, then show that

(*a*) there exists a continuous function on *E* which is not bounded;

(*b*) there exists a continuous and bounded function on *E* which has no maximum. 2+3

SECTION—B

(COMLEX ANALYSIS)

(20 Marks)

Answer ***all*** questions

All questions carry equal marks

1. Suppose that *f* is analytic in a domain Ω. If one of $| f |$, $\text{Re}(f)$ is constant, then show that *f* is constant on Ω. 2½+2½

2. (*a*) Prove that $\oint_c f(z)\,dz = 0$, if $f(z)$ is analytic with derivative $f'(z)$ which is continuous at all points inside and on a simple closed curve *C*.