MASTER OF SCIENCE EXAMINATION, 2018

(1st Year, 1st Semester)

MATHEMATICS

Unit - 1.3

(Complex Analysis)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Notations have their usual meanings.

Answer any *five* questions.

1. (a) Let a pair of functions $u(x, y)$ and $v(x, y)$ be defined in an open region R. If the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous and satisfy Cauchy Riemann equations in R then prove that the complex function $f(z)=u(x, y)+iv(x, y)$, $(z=x+iy)$ is holomorphic in R.

(b) Consider the function

$$f(z)=\begin{cases}0, \text{ if } (x, y) \text{ lies either on the } x\text{-axis or on the } y\text{-axis,}\\ 1, \text{ otherwise.}\end{cases}$$

[*Turn over*]

Does $f(z)$ satisfy Cauchy Riemann equation at $z=0$. Is the function $f(z)$ differentiable at $z=0$? Justify.

6+4

2. (a) Evaluate $\int_{\Gamma}\left(z-z^2\right)dz$, where $\Gamma = L(0,\ 3)+S(2,\ 1)$. Here $L(0,\ 3)$ is a line joining the points $z=0$ and $z=3$ and $S(2,\ 1)$ is a semicircle with center at $z=2$ and unit radius.

(b) Let Γ be a rectifiable closed Jordan curve. Then show that

$$\oint_{\Gamma}\frac{dz}{z-z_0}=\begin{cases}\pm\ 2\pi\ i,\ \text{if}\ \ z_0\in\ I(\Gamma)\\ 0,\ \text{if}\ \ z_0\in\ E(\Gamma)\end{cases}$$

4+6

3. (a) Let $f(z)$ be holomorphic in a simply connected open region R and Γ be positively oriented rectifiable closed Jordan curve such that $\Gamma \cup I(\Gamma)\subseteq R$. Prove that the values of the holomorphic function $f(z)$ on Γ uniquely determine its value at any point in $I(\Gamma)$.

(b) Let $f(z)$ be continuous in a simply connected open region R and $z_0 \in R$. If for every $z\in R$, the integral of $f(z)$ from z_0 to z is independent of the path, then show that the function $F(z)=\int_{z_0}^{z} f(\zeta)\,d\zeta$, is holomorphic in R and $F'(z)=f(z)$ in R.

(c) Define primitive of a function $f(z)$ defined in an open region R. 5+4+1

4. (a) Let $\{f_n(z)\}$ be a sequence of functions such that $f_n(z)\to f(z)$ uniformly on a rectifiable curve Γ and $f_n(z)$ is continuous on Γ for every n, then $\int_\Gamma f(z)$ exists and

$$\lim_{n\to\infty}\int_\Gamma f_n(z)\,dz=\int_\Gamma f(z)\,dz.$$

(b) If $\{f_n(z)\}$ be a sequence of functions holomorphic in a simply connected open region R and $f_n(z)\to f(z)$ uniformly on every compact set $E\subseteqq R$, then $f(z)$ is holomorphic in R and for every fixed positive integer k, $f_n^{(k)}\to f^{(k)}(z)$ uniformly on every compact set $E\subseteqq R$. 4+6

[*Turn over*]

5. (a) State Rouche's theorem.

(b) If $f(z)$ is holomorphic and non constant in an open region R, then prove that under the mapping $w = f(z)$, the image of R in the w-plane is also an open region.

(c) Show that the equation of the circle $K : |z - z_0| = r$ can be written as $\left|\frac{z-p}{z-q}\right| = \frac{|p - z_0|}{r}$, where p, q are inverse points with respect to the circle and $I(K)$ is given by $\left|\frac{z-p}{z-q}\right| < \frac{|p - z_0|}{r}$ if $p \in I(K)$. 1+5+4

6. (a) If $f(z)$ is holomorphic in an annulus $A(z_0, r_1, r_2)$, then show that

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n, \; z \in A(z_0, r_1, r_2)$$

where

$$a_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz, \; n = 0, \pm 1, \pm 2, \ldots$$

[*Turn over*]

and Γ is a positively oriented rectifiable closed Jordan curve such that $I(\Gamma)\cup\Gamma\subseteqq I(K_1)$, $I(K_2)\cup K_2\subseteqq I(\Gamma)$, K_1 and K_2 are circles $K_1(z_0, r_1)$ and $K_2(z_0, r_2)$ respectively.

(b) Let $f(z)$ be a meromorphic function in an open region R and Γ is a positively oriented rectifiable closed Jordan curve such that $I(\Gamma)\cup\Gamma\subseteqq R$. If $f(z)$ has neither zeros nor poles on Γ, then

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{f'(z)}{f(z)}\,dz = n-p\,,$$

where n and p denote respectively the number of zeros and number of poles of $f(z)$ in $I(\Gamma)$. 6+4

7. Let $f(z)$ be holomorphic in an open region R and $z_0 \in R$. Then prove that

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\,dz,\quad z\in N(z_0,\ r),$$

where $N(z_0, r)$ is the largest neighbourhood of z_0 contained in R.

[*Turn over*]

Hence deduce that $f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n\,dz$, for all $z\in\mathbb{C}$, for any entire function $f(z)$.

Hence prove that if $f(z)$ is an entire function and $|f(z)|\le M|z|^m$ for all $z\in\mathbb{C}$, where $M>0$ is a constant and *m* is a positive integer, then $f(z)$ is a polynomial of degree less than or equal to *m*. 5+2+3

8. Use the method of contour integration to prove the following results (any *two*) : 5+5

(a) $\int_0^{\infty}\frac{x^2}{x^4+a^4}dx=\frac{\pi}{2a\sqrt{2}},\quad a>0.$

(b) $\int_0^{\infty}\frac{x\sin x}{x^2+a^2}dx=\frac{\pi}{2}e^{-a},\quad a>0.$

(c) $\int_0^{2\pi}\frac{dt}{1-2a\cos t+a^2}=\frac{2\pi}{1-a^2},\quad |a|<1.$