Master of Science Examination, 2018

(2nd Year, 2nd Semester)

MATHEMATICS

Unit - 4.1

(Advanced Functional Analysis)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Symbols/Notations have their usual meanings.

Answer question no. 1 and any *three* from the rest.

1. Is a vector space endowed with the cofinite topology, a topological vector space ? Answer with reasons. 2

2. (a) Let X be a topological vector space. If K is a compact and C a closed subset of X with $K \cap C = \phi$ then show that there is a neighborhood V of θ such that $(K+V) \cap (C+V) = \phi$.

   (b) Prove that in a topological vector space X, every convex neighbourhood of θ contains a balanced convex neighborhood of θ.

[*Turn over*]

(c) Suppose V is a neighborhood of θ in a topological vector space X. If $0 < r_1 < r_2 < \ldots$ and $r_n \to \infty$ as $n \to \infty$ then show that $X = \bigcup_{n=1}^{\infty} r_n V$. 6+6+4

3. (a) Let Y be a locally compact subspace of a topological vector space X. Then prove that Y is a closed subspace of X.

(b) For a non-zero linear functional Λ on a topological vector space X, show that following are equivalent.

(i) Λ is continuous

(ii) The null space $N(\Lambda)$ is closed.

(iii) $N(\Lambda)$ is not dense in X.

(iv) Λ is bounded in some neighborhood V of θ. 8+8

4. (a) If $\{x_n\}_n$ is a sequence in a metrizable topological vector space X and $x_n \to \theta$ as $n \to \infty$ then prove that there is a sequence of $(+)$ve scalars $\{r_n\}_n$ such that $r_n \to \infty$ and $r_n x_n \to \theta$ as $n \to \infty$.

[*Turn over*]

(b) Define a bounded subset of a topological vector space X. Prove that $E \subset X$ is bounded iff for any sequence $\{x_n\}_n$ in E and a sequence of scalers $\{\alpha_n\}_n$ where $\alpha_n \to 0$ as $n \to \propto$, we have $\alpha_n x_n \to \theta$ as $n \to \propto$.

(c) Define the Minkowski functional μ_A of a convex absorbing set A. If A is a convex absorbing and balanced neighborhood of θ then show that μ_A is seminorm. 4+(1+6)+5=16

5. (a) If S is either (a) a complete metric space or (b) a locally compact Hausdorff space, then prove that the intersection of every countable collection of dense open subsets of S is dense in S.

(b) Let X and Y be topological vector spaces, and Γ be a collection of linear mappings of X into Y. Let B be the set of all $x \in X$ whose orbits $\Gamma(x) = \{\Lambda x : \Lambda \in \Gamma\}$ are bounded in X. If B is of second category in X then prove that Γ is equicontinuous.

(c) If Γ is an equicontinuous family of linear mappings from a topological vector space X into Y then show that f or any bounded subset E of X, there is a bounded subset F of Y such that $\Lambda(E) \subset F \;\; \forall \Lambda \in \Gamma$. 8+5+3=16

[*Turn over*]

6. (a) Suppose that

(i) M is a subspace of a real vector space X,

(ii) $p : X \to R$ satisfies $p(x+y) \le p(x) + p(y)$ and $p(tx) = t\,p(x)$ for $x, y \in X$ and $t \ge 0$,

(iii) $f : M \to R$ is linear and $f(x) \le p(x)$ on M then prove that there is a linear mapping $\Lambda : X \to R$ such that $\Lambda x = f(x)$ for $x \in M$ and
$-p(-x) \le \Lambda\, x \le p(x)\ \forall\ x \in X$.

(b) Prove that a non-constant linear functional Λ on a topological vector space X is open.

(c) Let A and B be disjoint nonempty convex sets in a topological vector space X. If A is open then prove that there is a $\Lambda \in X^*$ and $r \in R$ such that
Re $\Lambda\, x < r \le R_S\ \Lambda y$ for all $x \in A$ and $y \in B$.

7+3+6=16