BACHELOR OF SCIENCE EXAMINATION, 2018

(2nd Year, 1st Semester)

MATHEMATICS

Unit - 3.3

(Analysis I)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Symbols/Notations have their usual meanings.

1. (a) Show that the following are equivalent :

   (i) Every non-empty bounded above subset of R has a least upper bound.

   (ii) Let A, B be two non-empty disjoint subsets of R such that $R = A \bigcup B$ and $a < b$ for all $a \in A$, $b \in B$. Then there exists a unique element $c \in R$ such that $a \leq c \leq b$ for all $a \in A,\ b \in B$. 4+4

   (b) Give an example to show that (ii) may fail to hold in the system of rational numbers. 2

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2. (a) Let $A, B \supset R$. Then prove that

(i) *IntA* is the largest open set contained in *A*.

(ii) $Int(IntA) = IntA$.

(iii) Give an example to show that

$Int(A \cup B) \neq IntA \cup IntB$. 6

(b) Show that if S is a non-empty clopen subset of R then $S = R$. 4

3. (a) Show that every closed and bounded subset of *R* is compact. 6

(b) A subset *F* of *R* is compact iff every open cover has a finite subcover. Using the definition verify whether the following sets are compact or not :

(i) $(-2, 1]$;

(ii) *N*, the set of natural numbers. 4

4. (a) Let $f : R \to R$ be continuous on *R* and *D* be a dense subset of *R*. If

$f(x)=0 \ \forall x \in D$

then show that $f=0$ on R. 4

(b) Let $f: A \subset R \to R$ be continuous on A. If $\{x_n\}$ is a convergent sequence in A then show that $\{f(x_n)\}$ is also convergent. Verify whether 'convergent' can be replaced by 'Cauchy' ? 4+2

5. (a) Let f be defined and continuous on a closed set S in R. Let $A=\{x \in S : f(x)=0\}$. Then show that A is a closed subset of R. 4

(b) Let $f: R \to R$ be a continuous function such that $f(x)$ takes only rational values. If $f(1)=0$ then find $f(-1)$. 3

(c) Let $f: R \to R$ be continuous on R and $f(x_0)>c$ for some x_0 in R. Then show that there exists a *nbd.* U of x_0 such that $f(x)>c \ \forall x \in R$. 3

6. (a) Let $f:[a, b] \to R$ be monotone increasing and $c \in (a, b)$. Then show that $f(c-)$ and $f(c+)$ exists and

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$f(c-)=\sup\{f(x): a \le x < c\}$ and

$f(c+)=\inf\{f(x): c < x \le b\}$. 6

(b) Let D_f denote the set of discontinuities of a monotone increasing function defined on an interval *I*. Show that *f* can't have discontinuities of the 2nd kind and D_f is countable. 4

7. (a) Let $f:(a,\ b)\to R$ be continuous. Then *f* is uniformly continuous if $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exists finitely. 6

(b) Establish the convergence or divergence of the series whose *n*-th term is

(i) $\dfrac{1}{\left(n(n+1)\right)^{1/2}}$

(ii) $\dfrac{2^n}{e^n}$ 4

8. (a) If $\sum a_n$ is convergent with $a_n > 0$, then is $\sum \sqrt{a_n}$ convergent ? Either prove it or give a counterexample. 3

(b) Let $A, B \subset R$ and $A+B=\{a+b : a \in A, b \in B\}$. If A and B are closed then does it imply that $A+B$ is closed ? Either prove it or give a counterexample. 3

(c) Consider the alternating series

$$1-1/2+1/3-...$$

Show that the series is conditionally convergent. 4