Ex/M.Sc./Math/22/4.3/A2.1/2018

MASTER OF SCIENCE EXAMINATION, 2018

(2nd Year, 2nd Semester)

MATHEMATICS

Unit - 4.3 (A - 2.1)

(Advanced Algebra - II)

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Unexplained Notations, Symbols and terms
have their usual meanings

Part - I

1. Answer any *ten*. $1\times10=10$

   (a) $\mathbb{Z}$ is not a primitive ring — Justify.

   (b) Matrix ring over a division ring is semisimple — Justify.

   (c) Give an example of a ring which is semisimple but not primitive.

   (d) Any left quasiregular ring is a Jacobson radical ring — Justify.

[*Turn over*]

(e) If R is a left Artinian ring then $J(R) \subseteq P(R)$ — Justify.

(f) Suppose χ is the character of a finite degree representation of a finite group G over $\mathbb{C}$. Then $\chi(g)$ is a real number if g is conjugate to g^{-1} — Justify.

(g) Find the regular representation of $\mathbb{Z}_3$.

(h) Suppose $P : G \to GL_n(F)$ is a representation of the group G over the field F. How can you define a representation of G which is equivalent to P ?

(i) For an infinite dimensional vector space V over a division ring D, the ring End_D (V) is not simple — Justify.

(j) Every ideal of a semisimple ring is semisimple — Justify.

(k) $P(R) \subseteq J(R)$ — Justify.

(l) If F is a field with char $F = 2$ and V, W are vector spaces over F then for any multilinear form $f : V^n \to W$.

alternate $\Rightarrow$ symmetric $\Leftrightarrow$ skew symmetric

— Justify.

[*Turn over*]

Part - II

2. Answer any *two*. 10×2=20

(a) Classify the primitive rings in the class of commutative rings (not necessarily with identity).

(b) Prove that a simple ring with identify is primitive. Give an example (with explanation) to illustrate that the converse is not true. 4+(2+4)

3. (a) Prove the Jacobson Density Theorem on primitive rings.

(b) Suppose V_i is an n_i dimensional vector space over a division ring $D_i\,(i=1,2)$. Suppose the rings $\mathrm{End}_{D_1}(V_1)$ and $\mathrm{End}_{D_2}(V_2)$ are isomorphic. Prove that $\dim V_1 = \dim V_2$ and the division rings D_1 and D_2 are isomorphic. 4+6

4. (a) $J(R)$, the Jacobson radical of a ring *R* (not necessarily commutative, not necessarily with identity) is a both sided ideal — Justify.

(b) Prove that left quasiregularity is a radical property and that its associated radical is the Jacobson radical. 2+8

[*Turn over*]

Part - III

Answer any *two*. 10×2=20

5. (a) Suppose G is a group and F is a field. Prove that for any representation $\rho : G \to GL_n(F)$ there corresponds an FG-module and vice-versa.

(b) Consider the regular representation of $G = S_3$ over $F(= \mathbb{R} \text{ or } \mathbb{C})$. Let $V = \text{span}\{v_1, v_2, v_3\}$ be the corresponding FG-module and $U = \text{span}\{v_1 + v_2 + v_3\}$. Prove that U is an FG-submodule of V. Apply Maschke's theorem to find an FG-submodule W of V such that $V = U \oplus W$.

(c) The representation for $G = \{a : a^3 = e\}$ given by $\rho : G \to GL_2(\mathbb{C})$, $a\rho = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ is reducible — Explain. 3+4+3

6. (a) Let G be a finite group and $\rho : G \to GL_n(\mathbb{C})$ be a representation. Suppose g is an element of G of order m. Prove that there exists an ordered basis $\mathscr{B}$ of the $\mathbb{C}G$-module $\mathbb{C}^n$ such that the matrix $[g]_{\mathscr{B}}$ is a diagonal matrix with each diagonal element an m-th root of unity.

[*Turn over*]

(b) Suppose G is a finite group and χ, ψ are respectively characters of two finite degree representations of G over $\mathbb{C}$. Prove that $\langle \chi, \psi \rangle = \sum_{i=1}^{r} \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}$ where <,> is an inner product (to be suitably defined by you) on the vector space Maps $(G, \mathbb{C})$ and $g_1, g_2, \ldots, g_r$ are the representatives of the distinct conjugacy classes of G. Hence deduce the row orthogonality relations for the character table of G.

(c) If $\chi_1 \cdots, \chi_k$ are the characters of the inequivalent irreducible representations of a finite group G over $\mathbb{C}$ then they form a basis of vector space of class functions on G — Justify. 2+5+3

7. (a) Let R be a commutative ring with identity.

(i) Define an $n \times n$ determinant function on R.

(ii) Prove that there exists a unique determinant function on R.

(iii) The symmetric algebra of an R-Module M is a commutative ring — Justify.

(iv) The multiplication that defines the exterior (or wedge) product in the exterior algebra of an R-module is anticommutative on simple tensors — Justify.

[*Turn over*]

(b) Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/n\mathbb{Z}$. Then M is an R-module. Prove that the tensor algebra

$$\tau(M) \simeq \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \oplus \ldots \simeq \mathbb{Z}[x]/\langle nx \rangle.$$

(1+3+1+2)+3