Ex/UG/SC/Core/Math/TH/03/2018(New)

Bachelor of Science Examination, 2018

(1st Year, 2nd Semester)

MATHEMATICS (Honours)

Unit - Core 3

(Real Analysis)

[New Syllabus]

Full Marks : 50 Time : Two Hours

The figures in the margin indicate full marks.

Use a separate Answer sheet for each Group.

Symbols and Notations have their usual meanings.

Group - A

(Marks - 25)

Answer any *five* questions.

1. Prove that the set of rational numbers is dense in the set of real numbers. Hence show that the set of irrational numbers is also dense in the set of real numbers. 4+1

[*Turn over*]

2. (i) Using the least upper bound property of the real numbers show that every closed interval $[a,b]$, $a<b$ is compact.

   (ii) Consider $S=\{x\in Q:1\le x\le 2\}$. Justify whether S is compact in Q. 3+2

3. Define derived set of a set. Give an example of a set A for which $\left((A')'\right)' = \emptyset$ but $(A')' \neq \emptyset$. 2+3

4. Define closure of a set. Let x be a real number and $A\subset R$. Show that $x\in cl\,A$ iff $d(x,A)=0$ where $d(x,A)=\inf\{|x-a|:a\in A\}$. 2+3

5. Prove that *Int A* is the largest open set contained in *A*. Give an example to show that $Int(A\cup B)\neq Int\,A\cup Int\,B$. 3+2

6. (i) Let F be an Archimedean ordered field. Show that if F satisfies lub property then F satisfies Cantor's nested interval property.

   (ii) Justify whether the set of rational numbers satisfy Cantor's nested interval property ? 3+2

[*Turn over*]

7. A subset F or R is compact iff every open cover has a finite subcover. Using the definition verify whether the following sets are compact or not : (i) $(-2,1]$; (ii) N, the set of natural numbers. 3+2

8. Let $\{F_n\}$ be a sequence of non-empty closed sets of real numbers with $F_{n+1} \subset F_n$. Show that if F_1 is bounded then $\bigcap_{n=1}^{\infty} F_n \neq \phi$. Show that the conclusion may fail if we do not assume that F_1 is bounded. 3+2

Group - B

(Marks - 25)

Answer Question No. 9 and any *two* from the rest.

9. (a) Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ converges to $\log 2$ but the rearranged series $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots$ converges to $\frac{3}{2}\log 2$.

   (b) Give an example of an unbounded sequence that has both convergent and divergent subsequences. Justify your answer. 3+2

10. (a) Show that the sequence $\left\{\left(1+\frac{1}{n}\right)^{n}\right\}$ is a monotone increasing and bounded. Is the sequence convergent ? Justify your answer.

(b) Prove that the series $\sum_{n=1}^{\infty} n^{\frac{1}{p}}$ is divergent if $p \leq 1$ (not using integral test).

(c) An absolutely convergent series is convergent. 5+3+2

11. (a) Prove that a bounded sequence of real numbers has a convergent sub-sequence.

(b) Find the region of convergence of the series

$$x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\ldots$$

5+5

12. (a) Show that $\left\{\frac{\cos n\pi}{n}\right\}$ is a Cauchy sequence.

(b) Prove that every convergent sequence is bounded and its limit is unique.

(c) Test the convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^2+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^3+\ldots$$

2+4+4