BACHELOR OF SCIENCE EXAMINATION, 2018

(3rd Year, 2nd Semester)

MATHEMATICS (Honours)

Paper - 6.4

(Analysis - IV)

Full Marks : 50 | Time : Two Hours

The figures in the margin indicate full marks.

Use a separate Answer-script for each part.

Part - I

(Marks - 30)

Answer any *three* questions.

1. (a) When a function $f : R^n \to R^m$ is said to be differentiable at $\tilde{a} \in R^n$? If $f : R^n \to R^m$ is differentiable at $\tilde{a} \in R^n$ then show that f is continuous at $\tilde{a}$ and all the directional derivatives of f exist at $\tilde{a}$. 5

(b) Let $f(x,y) = x + y$, $x = 0$ or $y = 0$

$= 1$, otherwise

Show that the partial derivatives exist at (0,0) but the other directional derivatives fail to exist. 2

[*Turn over*]

(c) Let $f(x,y) = \dfrac{xy^2}{x^2+y^4}$, $x \neq 0$

$= \quad 0 \quad , \; x = 0$

Show that the directional derivatives exist at $(0,0)$ but f is not continuous at $(0,0)$. 3

2. (a) Suppose $f : R^n \to R$ be such that one of the partial derivatives exist at $\tilde{a} \in R^n$ and the remaining $(n-1)$ partial derivatives exist in some open fall $B(\tilde{a}, r)$ and are continuous at $\tilde{a}$. Then show that f is differentiable at $\tilde{a}$. 6

(b) Let $f(x,y) = \dfrac{x|y|}{\sqrt{x^2+y^2}}$, $(x,y) \neq (0,0)$

$= \quad 0 \quad , \; (x,y) = (0,0)$

Show that all the directional derivatives of f exist at (0,0) but f is not differentiable at (0, 0). 4

3. (a) State and prove Schwarz's Theorem. 5

(b) Let $f : R^2 \to R^2$ be defined by $f(x,y) = (e^x \cos y,\; e^x \sin y)$ $\forall\, (x,y) \in R^2$. Show that f is one-one on the set A consisting of all (x, y) with

[*Turn over*]

$0 < y < 2\pi$. If g is the inverse function defined on $f(A)$ then find $Dg(0,1)$. 5

4. (a) Let $f : S \to R^m$ be differentiable at each point of S where S is an open connected subset of R^n. If $f'(\tilde{c}) = 0$ for each $\tilde{c} \in S$ then prove that f is constant on S. 3

(b) Let $f : R^n \to R^n$ has continuous partial derivatives $D_j f_i$ (for $i = 1, 2, ...n$, $j = 1, 2, ...n$) on an open set $S \subset R^n$ and $J_f(\tilde{a}) \neq 0$ for some point $\tilde{a} \in S$. Then prove that there is an open ball $B(\tilde{a}, r)$ on which f is one-one. 7

5. (a) Find the tangent and normal space at $(\sqrt{2}, \sqrt{2}, 1)$ to the set of points satisfying $x^2 + y^2 - 2z^2 = 2$ and $xyz = 2$. 4

(b) Find the shortest distance from the origin to the curve $x^2 + 8xy + 7y^2 = 225$, $z = 0$ and the points of shortest distance. 6

Part - II

(Marks - 20)

(Unexplained Notations and Symbols have their usual meanings.)

Answer Q. No.1 and any *three* from the rest.

1. Answer any *five* : 1×5=5

(a) For any $z \in \mathbb{C}_\infty$ the cross ratio $(z, 1, 0, \infty) = z$ and for any three distinct points z_2, z_3, z_4 of $\mathbb{C}_\infty$ the cross ratio $(z_4, z_2, z_3, z_4) = \infty$ — Justify. .

(b) Suppose $\{z_2, z_3, z_4\}$ and $\{w_2, w_3, w_4\}$ be two triple of distinct points of $\mathbb{C}_\infty$. If the mapping $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is unique with $sz_i = w_i$ (i=2,3,4) then (z_1, z_2, z_3, z_4) $= (sz, w_2, w_3, w_4)$ — Justify.

(c) Let $a, b \in \mathbb{C}$ with $|b| = 1$. What does the following set represent geometrically ?

$$\left\{Z \in \mathbb{C} : \text{Im}\,\frac{z}{b} > 0\right\}$$

[*Turn over*]

(d) Suppose $f : \mathbb{C} \to \mathbb{C}$ is a differentiable function such that $f(z) = u(x, y) + iv(x, y)$ for all $z = x + iy \in \mathbb{C}$. What is $f'(z)$ in terms of partial derivatives of u and v?

(e) Find a mapping $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ which maps the real axis to the unit circle.

(f) Find the geometrical significance of the mapping $f : \mathbb{C} \to \mathbb{C}$, defined by $f(z) = 2e^{\frac{i\pi}{4}} z$.

2. Let $\varphi : S^2 \to \mathbb{C}_\infty$ and $\psi : \mathbb{C}_\infty \to S^2$ denote the stereographic projection and its inverse.

(a) Define a metric d on $\mathbb{C}_\infty$ which is preserved by the stereographic projection. Obtain the restriction on the real number R for this to be the radius of a neighbourhood of ∞ in $\mathbb{C}_\infty$ with respect to d. Also find for such R the neighbourhood $N_d(\infty; R)$.

(b) Suppose $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is defined by $f(z) = \frac{1}{z}$ for all $z \in \mathbb{C}_\infty$. Determine $\hat{f} : S^2 \to S^2$ such that $\hat{f} \circ \psi = \psi \circ f$. Find the geometrical significance of $\hat{f}$. 3+2

[*Turn over*]

3. (a) For every complex function $f:\mathbb{C}^2 \to \mathbb{C}^2$, defined by $f(z)=u(x,y)+iv(x,y)$ for all $z=x+iy\in\mathbb{C}$, there corresponds a function $F:\mathbb{R}^2 \to \mathbb{R}^2$, defined by $F(x,y)=(u(x,y),v(x,y))$ for all $(x,y)\in\mathbb{R}^2$. Prove that for $z=x+iy\in\mathbb{C}$, if $f'(z)$ exists then $F'(x,y)$ exists. Provide an example to illustrate that the converse is not true.

(b) Let $u(x,y)=x^3-3xy^2-2y$ for all $(x,y)\in\mathbb{R}^2$. Verify that u is harmonic and determine a function $v:\mathbb{R}^2\to\mathbb{R}$ such that $f=u+iv$ is analytic. 3+2

4. (a) For real A, B, C, D with $A^2+B^2+C^2>D^2$, what is the geometrical meaning of

$$A(\overline{z}+z)+Bi(\overline{z}-z)+C\left(|z|^2+1\right)+D\left(|z|^2-1\right)=0 \quad ?$$

Suppose Γ_1 is a circle in C and Γ_2 is the projection of Γ_1 under the stereographic projection. Prove that Γ_2 is a circle not passing through the north pole.

(b) Suppose $f(z)$ and $\overline{f(z)}$ both are analytic on an open connected set D of $\mathbb{C}$. Prove that f is constant. 4+1

[*Turn over*]

5. (a) Suppose Γ_1 and Γ_2 are two circles in $\mathbb{C}_\infty$. Prove that there exists a Möbius transformation $M:\mathbb{C}_\infty \to \mathbb{C}_\infty$ such that $M(\Gamma_1)=\Gamma_2$. Also prove that there are more than one such Möbius transformation.

(b) Let $f:\mathbb{C}\to\mathbb{C}$ be defined by $f(z)=z^2$. Show that f is not conformal at $z=0$ by considering the angle between the line $\theta=\frac{\pi}{4}$ and the real axis and the angle between their images. Does this contradict the result on conformality of an analytic function defined over a connected open set of $\mathbb{C}$? 3+2

6. (a) Let $G=\{Z\in\mathbb{C}:\operatorname{Im} z>0\}$ and $D=\{z\in\mathbb{C}:|z|<1\}$. Find an analytic function $f:G\to\mathbb{C}$ that maps G to D.

(b) Comment on the validity of the statements $e^{\log z}=z$, $\log e^z=z$ for any complex number z (log denotes the principal value). 4+1