(b) Consider the equation :

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+\left\{K+\frac{2}{x}-\frac{l(l+1)}{x^2}\right\}y=0,\ 0\le x\le\infty$$

where l = a non-negative integer and $y = y(x)$.

(A similar equation arises in solving the Schrödinger equation for the hydrogen atom). Find *all* values of the constant K which can give a solution which is *finite* (*i.e.*, does not blow up/diverge) on the entire range of x (including ∞). [6+4]

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M. Sc. Examination, 2025

[1st Year, 1st Semester (Day)]

PHYSICS

Paper : PHY/PG/CORE/TH/102

(Mathematical Methods)

Time : 2 *Hours* *Full Marks* : 40

Put corresponding question numbers against your answer. The symbols used below will carry their usual meanings, unless otherwise stated.

Use a separate Answer Script for each Group

Group—A

1. Diagonalise σ_x by the means of a similarity transformation. Show that the trace of an operator is independent of the choice of basis. Argue from here that the trace of a matrix must equal to the sum of its eigenvalues. 5 **(CO3)**

(OR)

Prove the Cauchy-Schwartz and the triangle inequalities for an inner product space. 5 **(CO3)**

2. For a general inertial transformation, $x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu}$, prove that

$$\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}$$

For an infinitesimal Lorentz transformation, $\Lambda^{\mu}_{n} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}$ show that $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Consider now an infinitesimal $t - x$ boost. Find out the corresponding $\omega_{\mu\nu}$. 5 **(CO5)**

(OR)

Show that, $\partial_{\mu}F^{\mu\nu} = -j^{\nu}$ implies continuity equation. Find out the relevant Maxwell's equations from the above equation. 5 **(CO5)**

3. *(a)* Consider a function $f(z) = |z|$ on the Argand plane. Is it analytic? Show that the integral

$$\int_{-\infty}^{\infty} \frac{dx\, e^{ipx}}{x - i\varepsilon},$$

where p is negative and real and $\varepsilon = 0^{+}$, is vanishing.

(b) Evaluate the following integrals using contour integration method.

(i) $\int_{0}^{\infty} dx\, e^{ix^2}$ *(ii)* $\int_{0}^{\infty} \frac{dx\, x^{\alpha-1}}{x^2 + 4}$

where for the second integral, $0 < \alpha < 2$.

(1+2) **(CO2)** + (3+4) **(CO2)**

(b) The Bessel's equation for $m = 0$, is given by

$$x^2 y'' + xy' + x^2 y = 0$$

where $y = y(x)$, $y' = \frac{dy(x)}{dx}$, etc. It is known that one solution to this equation is given by :

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots$$

Show that the second solution is of the form :

$$J_0(x)\ln x + Ax^2 + Bx^4 + Cx^6 + \cdots$$

and hence find the first three coefficients A, B, C.

[4+6]

(OR)

4. *(a)* Consider the heat flow problem for an infinite one-dimensional solid conductor, with a given initial temperature distribution (the "heat source"). The applicable equation for the temperature distribution function $T(x,t)$ in this case is

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{k}\frac{\partial T(x,t)}{\partial t}, \text{ with } T(x,t) = f(x)$$

Find the general expression for the temperature distribution function $T(x,t)$ inside the solid for all future times $t > 0$. **[CO1]**

(OR)

Take a function $f(z)=u(x,y)+iv(x,y)$. Show that if $f(z)$ is analytic, then both u and v satisfy the Laplace equation. Evaluate the following integrals using suitable contours.

(i) $PV\int_{-\infty}^{\infty}\frac{dx}{x-1}$ *(ii)* $\int_{0}^{\pi}\frac{\alpha\, d\phi}{\alpha^2+\sin^2\phi}$,

where α is real and positive.(2) **(CO2)** + (4+4) **(CO2)**

Group—B

1. *(a)* Consider the following differential equation :

$$\frac{d^2y}{dx^2}+xy=0$$

Show how the strategy of using integral transforms can be used to provide a quick solution to the above equation. (You may leave your answer in the form of a definite integral). **[CO4]**

(b) Consider the following differential equation, a second order *ODE*, for the function $\psi(x)$:

$$\frac{d^2\psi(x)}{dx^2}+\lambda\psi(x)=f(x)$$

where $\psi(x)\to 0$, in the limit $|x|\to 0$ and $f(x)$ is a given smooth (real) function and λ is a constant. Define the Green's function for the above equation and hence obtain the general solution $\psi(x)$ in terms of it. **[4+6]**

(OR)

2. *(a)* Three radioactive nuclei decay successively in series, so that the numbers $N_k(t)\,(k=1,2,3)$ of the three types at any time t obey the equations :

$$\frac{dN_1}{dt}=-\lambda_1 N_1$$

$$\frac{dN_2}{dt}=\lambda_1 N_1-\lambda_2 N_2$$

$$\frac{dN_3}{dt}=\lambda_2 N_2-\lambda_3 N_3$$

If initially (*i.e.*, at $t=0$), $N_1=N,\ N_2=0,\ N_3=n$, find the expression for $N_3(t)$.

(b) Given two functions $f_1(x)$ and $f_2(x)$, their *convolution* $*$ is defined by :

$$g(x):=(f_1*f_2)(x)=\int_{-\infty}^{\infty}ds\ f_1(s)\,f_2(x-s)$$

Show that the Fourier transform $\mathcal{F}$ for the convolution satisfies the simple product rule :

$\mathcal{F}[g(x)]=\mathcal{F}[f_1(x)]\cdot\mathcal{F}[f_2(x)]$ (up to a constant)

3. *(a)* Consider the general *ODE* second order in the following standard form :

$$\frac{d^2y(x)}{dx^2}+P(x)\frac{dy(x)}{dx}+Q(x)y(x)=0,$$

where $P(x)$ and $Q(x)$ are rational functions of x. Define the notion of an *ordinary* point and a *regular singular* point in the context of the above equation. By making a suitable change of variable or otherwise, obtain/state the conditions for the point $x=\infty$ to be a regular singular point of the above equation.