M. Sc. Physics (Day) Examination, 2024

(1st Year, 1st Semester)

Physics

Paper : PHY/PG/CORE/TH/102

(Mathematical Methods)

Time : 2 *Hours* *Full Marks* : 40

(Use separate answer script for each Group.)

(The symbols used below will carry their usual meaning, unless otherwise stated)

GROUP—A

4×5=20

1. Define a linear vector space. What is meant by the linear independence of vectors? Show that if two vectors are mutually orthogonal, they must be linearly independent. Prove that the converse is not necessarily true. **[5(CO3)]**

(OR)

Prove that the trace of an operator is independent of the choice of basis. Show also that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Let us consider a unitary operator *U*. Show that its eigenvalues must be of the general form: $e^{i\phi}$, where ϕ is real. Consider a similarity transformation, $U^{\dagger}\sigma_x U = \sigma_z$. Find out *U*.

[5(CO3)]

2. Show that for any Lorentz transformation matrix $\Lambda_\mu{}^\nu$, we must have $\eta_{\alpha\beta}\Lambda_\mu{}^\alpha\Lambda_\nu{}^\beta = \eta_{\mu\nu}$. For an infinitesimal Lorentz transformation, the transformation matrix can be splitted as $\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \omega_\mu{}^\nu$, where $|\omega| \ll 1$. Show that $\omega_{\mu\nu}$ is anti-symmetric in its two indices. State the principle of general covariance. **[5(CO5)]**

(OR)

Consider the Bianchi identity : $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$. Show that this implies the Maxwell equations, $\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$. Under what circumstances this identity will be violated? **[5(CO5)]**

3. Let $G(x,x')$ be the Green function for the Klein-Gordon differential operator $\left(\partial_\mu\partial^\mu - m^2\right)$ in d-spacetime dimensions. Find out the length dimension of $G(x,x')$. Consider now $d = 4$. Show that the quantity,

$$iG(x,x') = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\left(\theta(t-t')e^{-ik\cdot(x-x')} + \theta(t'-t)e^{ik\cdot(x-x')}\right)$$

is the Green function for the Klein-Gordon operator (take $k^o = \omega_k = \sqrt{\vec{k}^2 + m^2}$). **[1+4(CO2)]**

3. Solve the following partial differential equation for $P(x,t)$:

$$\frac{\partial P(x,t)}{\partial t} = -D_1\frac{\partial P(x,t)}{\partial x} + D_2\frac{\partial^2 P(x,t)}{\partial x^2}$$

(where D_1 and D_2 are positive constants), with the given initial condition :

$$P(x,0) = \frac{1}{\sigma\sqrt{\pi}}e^{-x^2/2\sigma^2}$$ **10**

4. *(a)* Find the lowest frequency of oscillation of acoustic waves ψ inside a hollow sphere of radius *R*. The waves obey differential equation :

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

along with the boundary condition: $\left.\left(\frac{\partial\psi}{\partial r}\right)\right|_{r=R} = 0$

(b) Write down all the values of i^i in the form of $a + ib$. **(8+2)**

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GROUP—B

(Answer ***any*** two questions)

[(CO1) & (CO2)]

1. *(a)* If *f* is holomorphic in a neighbourhood of z_0, and z_0 is a *zero* of *f* of order *m*, then prove that

$$\text{Residue } \left(f'/f\right)\big|_{z=z_0} = m .$$

(b) Consider the following differential equation for the function $\psi(x)$:

$$\frac{d^2\psi(x)}{dx^2} + A\psi(x) = f(x)$$

where $\psi(x)$ vanishes in the limit $|x| \to 0$. Define the Green's function for the above equation and express the general solution $\psi(x)$ in terms of it. Hence explain how the method of Green's function works for solving inhomogeneous PDEs. **(3+7)**

2. *(a)* Compute the residue(s) of the following function at all its poles, including infinity. $(n \in Z^+)$,

$$f(z) = \left(\frac{z^{n-1}}{z^n - 1}\right)$$

(b) Let $f(z) = \dfrac{z^5}{|z|^4}$ $(z \neq 0)$, and $f(0) = 0$. Show that the real and imaginary parts of *f* satisfy the Cauchy-Riemann equations at $z = 0$. Is the function *f* analytic at $z = 0$? Justify your answer. **(5+5)**

(OR)

Take in one dimension, $\dfrac{d^2G(x,x')}{dx^2} = \delta(x - x')$

Prove that the first derivative of $G(x,x')$ is discontinuous across x'. Consider the Sturm-Liouville type differential equation, $\mathcal{L}y = f(x)$, where $\mathcal{L}$ is the differential operator with respect to *x*. Find out an expression for the corresponding Green function using the eigenvalues and orthonormal set of eigenfunctions of $\mathcal{L}$. **[1·5+3·5(CO2)]**

4. Evaluate the integral : $\int_0^\infty ds\, e^{-ips}$ $(p > 0)$.

Draw the appropriate contour. Consider a rotation by an angle ϕ about the *z*-axis. Write down the rotation matrix. Do such matrices representing rotations around *z*-axis form a group? Explain. **[2(CO2)+3(CO4)]**

(OR)

Determine the inverse Fourier transform of $1/k^2$ in three dimensions, where $k^2 = k_x^2 + k_y^2 + k_z^2$. Consider the $t - x$ -Lorentz boost. Write down the corresponding transformation matrix and find out the generator. **[2(CO2)+3(CO4)]**