9. *(a)* [**CO-4**] In the context of discrete distribution, the Binomial distribution is, perhaps, the most common and important distribution. Its probability density function (pdf) is given by

$$f(x) = {}^{n}C_{x} p^{x} q^{n-x},$$

signifying the probability of obtaining x successes out of a total n trials, with p describes the probability of success and $q = (1 - p)$ is the probability of failure.

(i) Prove that, for a Binomial distribution $\text{var}(ax + b) = a^2 npq$, where 'var' denotes the variance and, a and b are some constants. Also mention the significance of variance in statistics.

(ii) Mention the conditions under which the Binomial distribution may tend towards Poisson distribution whose pdf is given by

$$f(x) = \frac{\lambda^{x} e^{-\lambda}}{x!}, \quad \text{where } \lambda = np.$$

(iii) Consider there are 1500 people in an auditorium. Each people select a number at random between 1 to 500. Now we would like to calculate the probability that two people select the same number, say, 81. By using the Poisson distribution, show that the probability $P(x = 2) \simeq 0{\cdot}2240$.

Next, if one applies the Binomial distribution formula in such a problem, then verify that, the probability $P(x = 2) \simeq 0{\cdot}2241$, which is almost identical.

[(3+1)+2+(1+1)][**CO 4**]

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Ex/SC/PHYS/UG/MDC/TH/11/101A/2024

B. Sc. PHYSICS EXAMINATION, 2024

(1st Year, 1st Semester)

PHYSICS

PAPER : UG/MDC/TH/11/101A

(Mathematical Methods in Natural Science-I)

Time : *Two Hours* *Full Marks* : 40

The figures in the margin indicate full marks.
Candidates are instructed to give their answers in their own words as far as practicable.

Answer ***any one*** from Q. **1** and Q. **2**

1. **[CO-1]** The one-dimensional wave equation is often considered to be an important second order linear partial differential equation and it reads as

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}, \text{ with } \psi \equiv \psi(x, t),$$

where v denotes the (phase) velocity of the wave.

(i) Choose the correct option :

The above wave equation belongs to the class of

(a) hyperbolic equations

(b) parabolic equations

(c) elliptic equations

(d) None of *(a)*, *(b)* and *(c)*

(ii) Verify that $\psi(x, t) = Ae^{-(x-vt)^2}$ satisfies the above wave equation, where A is a constant. Also mention the physical significance of such a solution.

(iii) Next, without no loss of generality, one may set $A = 1$. Moreover, let us introduce a new variable ζ such that $\zeta = (x - vt)$. Therefore, $\psi(\zeta) = e^{-\zeta^2}$. We further introduce a function $\chi(\zeta) = \zeta^2\psi(\zeta)$.

Roughly draw a curve $\chi(\zeta)$ *vs* ζ. Also verify that the total area under the curve of $\chi(\zeta)$ is $\frac{\sqrt{\pi}}{2}$, i.e., $\int_{-\infty}^{\infty} \chi(\zeta) d\zeta = \frac{\sqrt{\pi}}{2}$.

Given that $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx, \;\; n > 0,$ with

$\Gamma(n+1) = n \;\; \Gamma(n)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

[1+(3+1)+(1+3)][CO1]

2. **[CO-1]** The celebrated Laplace equation in two-dimensional Cartesian coordinate system reads as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \text{ with } u \equiv u(x, y).$$

It is easy to verify that a possible solution of the Laplace equation can be written as $u(x, y) = \ln(x^2 + y^2)$.

(i) Check whether the function $u(x, y) = \ln(x^2 + y^2)$ is a homogeneous function.

6. *(a)* Consider a relativistic particle under the action of a force that is proportional to the square of the time. Find the dependence of the velocity of the particle on time.

(b) What happens to the magnitude of the velocity in the long time limit? (7+2=9)**[CO 3]**

7. *(a)* Using the method of integrating factors, solve the differential equation $y' + y = e^x$ for the boundary condition $y(0) = 0$.

(b) A harmonic oscillator of unit frequency is being excited with a time dependent force $F(t) = t^3$. Determine the time-dependence of the displacement (from the mean position), if at the initial instant the oscillator starts from the mean position with unit velocity.

(3+6=9)**[CO 3]**

Answer ***any one*** from Q. **8** and Q. **9**.

8. *(a)* Find the Fourier series of the function defined as $f(x) = 0$ for $-\pi \le x < 0$ and $f(x) = \pi$ for $0 \le x \le \pi$.

(b) Hence find the value of the infinite series given by :

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots .$$

(6+2=8)**[CO 4]**

Hence, by employing the Euler's theorem on homogeneous function, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2$$

(ii) We now switching to the two-dimensional polar coordinate system by the relation : $x = r\cos\theta$ and $y = r\sin\theta$. By applying the chain rule of partial differentiation, we have found that

$$\frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta},$$

$$\frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

Prove that the Laplace equation in the two-dimensional polar form can be expressed as

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0$$

(ii) Next, by introducing a separation 'ansatz' $u(r,\theta)=R(r)\Theta(\theta)$, show that the radial part of the Laplace equation in polar form can be obtained as

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\lambda^2 R=0,$$

where λ is a separation constant. Hence obtain a general solution of the above radial-part equation.

[(1+2)+3+(1½+1½)][CO1]

Answer ***any two*** questions from Q. **3** – Q. **5** and ***any one*** from Q. **6** and Q. **7**.

3. *(i)* Find the directional derivative of $\phi = 4e^{2x+y+z}$ at the point (1, 1, –1) in direction towards the point (–3, 5, 6).

(ii) Use divergence theorem to show that

$\iint_s \nabla(x^2+y^2+z^2).\,d\overline{s}=6\,V$, where s is any closed surface enclosing volume V. (4½+2½)[CO 2]

4. *(i)* Define *base vector* $(\vec{b}_i)$ of any curvilinear coordinate system. Hence determine the unit vectors $(\hat{e}_i)$ of the spherical polar coordinate system. Identify the directions of the unit vector **or** the corresponding reference planes.

(ii) If $\phi\,(x, y, z)$ be a continuous scalar function, find out the mathematical expression of $\overline{\nabla}\phi$ in any orthogonal curvilinear coordinate system. (5+2)**[CO 2]**

5. *(i)* If $\lambda_1, \lambda_2, \lambda_3, \ldots.\lambda_n$ be the eigenvalues of a matrix A, show that the eigenvalues of A^{-1} are $\frac{1}{\lambda_1},\frac{1}{\lambda_2},\frac{1}{\lambda_3}........\frac{1}{\lambda_n}$.

(ii) Using the properties of the eigenvalues of a matrix, find out the **sum and product** of the eigenvalues of

$$A=\begin{pmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{pmatrix}$$

(iii) Define Hermitian matrix. Show that the eigenvalues of Hermitian matrix are real. (2+2+3)**[CO 2]**