(b) If F_{ij} is an anti-symmetric tensor and the quantity S_{ijk} vanishes, where

$$S_{ijk} := \frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} (= 0).$$

Show that the trivial solution for this would be the case that the tensor F_{ij} be of the form $F_{ij} := \partial_i A_j - \partial_j A_i$. Translate the above wisdom in the language of differential forms and in particular comment, on why the equation becomes merely an identity. [6+6]

★ ★ ★

Bachelor of Science Examination, 2024

(3rd Year, 1st Semester)

Physics

Paper : DSE/01/A1

(Advanced Mathematical Methods)

Time : *Three Hours* *Full Marks* : 75

*Answer from **both** the Groups as directed.*

The figures in the margin indicate full marks.

GROUP—A

Answer ***any three*** of the following questions : 13×3=39

1. Consider the symmetry transformations of a rectangle, drawn with its centre at the origin of x-y plane and parallel to x-axis, giving rise to the D_2 symmetry group.

(i) Identity transformation or no transformation at all is denoted as 'e'.

(ii) Reflection about vertical y-axis say (1, 3) is denoted as 'a'.

(iii) Reflection about the horizontal x-axis say (2, 4) is denoted as 'b'.

(iv) Rotation of the rectangle in the plane around the centre by π is denoted as 'c'.

(a) Construct the multiplication table.

(b) Find the two-dimensional and three-dimensional matrix realization of D_2 symmetry group.

(c) Find also the regular representation of D_2 symmetry group.

(d) Is there any one-dimensional representation of D_2 symmetry group?

(e) Construct the character table for all these representations. 13

2. Suppose a group G is given, it has a representation D_1 on linear vector space V_1 and another representation D_2 on V_2. Both are irreducible representations. Assume that V_1 and V_2 are complex liner vector spaces. Suppose one can construct a linear transformation $T : V_1 \to V_2$ such that the equation $TD_1(g) = D_2(g)T$ is true for all geG.

(i) If $T = 0$, can you draw any conclusion on the relationship between D_1 and D_2?

(ii) Consider $T \neq 0$ but T is singular/non-singular. Discuss about the possibility of such situation.

(b) Three of the most useful identities in vector calculus are the Green's Theorem, Gauss' Theorm and the Stokes' Theorem and each of these can be re-written succinctly as rather trivial identities. Using the language of differential forms; show how. [6+6]

7. *(a)* Now consider a special/shorter case of the problem of calculus of variations where the functional F does not depend on y. Thus, consider the modified variational problem of optimizing

$$I = \int_i^f dx\, F(y', x), \text{ where } y = y(x),\ y' = \frac{dy}{dx}.$$

Obtain the special form of the Euler-Lagrange equation in this case and show that it can be integrated readily. (You need not start from the first principle).

(b) A geodesic on a given surface is a curve, lying completely on that surface and along which, the distance between two points is the shortest. (For example, on the plane $\mathbb{R}^2$, a geodesic is a straight line). Now determine the equation of the geodesic on the surface of a sphere of radius R, that is for $S^2 : \{(x, y, z) : | x^2 + y^2 + z^2 = R^2\}$ [5+7]

8. *(a)* Expressing quantities in the language of differential forms, write a note on the generalized form of the Stokes' Theorem.

(iii) Define Null space of T i.e., $N(T)$ in V_1. Is it a subs-space of V_1?

(iv) Let $x \in V_1$ and also $x \in N(T)$, then show that if the group is allowed to act on x, then it remains on $N(T)$. 13

3. *(i)* Show that the set of translation $T = T(a)$ with the parameter of translation $a \in R$ (namely, $-\infty < a < \infty$) satisfies all the properties for defining a group. You may identify $T(a) = I + a\dfrac{d}{dx}$.

(ii) The set of translations is an example of continuous group and also abelian. Explain it.

(iii) If the parameters of translation were restricted to a finite interval between L_1 and L_2, then the set of translation will not define a group. Explain it.

(iv) Find the generator of translation in three-dimension.

13

4. *(a)* The rotation groups in the Euclidean plane, proper and improper, are defined as : $SO(2)/O(2) = A = 2{\times}2$ real matrix $A^TA = I_{2\times3}$, $det\ A = 1/\pm 1$. Show that $SO(2)$ is abelian, $O(2)$ non-abelian.

(b) Show that from a direct product $G_1 \times G_2$ of two groups, the individual factors can be recovered as invariant subgroups.

(c) For the permutation group S_n, review the following :

(i) Composition law, identity, inverses

(ii) Cycle structure of a permutation

Show that S_2 is abelian, S_n for $n \geq 3$ is non-abelian. 13

GROUP—B

Answer ***any three*** of the following questions : 12×3=36

5. *(a)* Using the ideas from Calculus of Variations, formulate the variation of the functional

$$I = \int_i^f dx\, F(y, y', x), \quad \text{where } y = y(x),\ y' = \frac{dy}{dx}.$$

Hence obtain the Euler-Lagrange equation that solves the optimization problem.

(b) Consider the centuries old *brachistochrone* problem of a rigid wire stretched between two poles of unequal vertical heights and a certain horizontal distance apart. A light bead is to slide smoothly along the wire from end to end under the influence of gravity. What should be the shape of the wire so that the journey takes the minimum time? [6+6]

6. *(a)* Consider the simple space $X = \mathbb{R}^4$. By enumerating/listing the various kinds of 2-forms and 3-forms that can live on X, find the dimensions of the space of 2-forms and 3-forms.