M.Sc. (Physics) 1st Yr, 1st Semester Examination, 2018.

PHY/TG/102: MATHEMATICAL METHODS– I

Max. Marks = 40. Max. Time = 2 hrs.

ANSWER Q.1 AND ANY TWO FROM THE REST:

(1). (a) Evaluate $\oint_\Gamma \bar{z}\, dz$ and $\oint_\Gamma z^{\frac{1}{2}}\, dz$ (consider the Principal Branch of $z^{\frac{1}{2}}$) in the two cases, when: (i) Γ is the circle $|z| = 1$, and (ii) Γ is the circle $|z-1| = 1$.

(b). Find the first few terms in the Laurent expansion of $1/(\cos z - 1)$ valid for the regions: (i) $|z| < 2\pi$, and (ii) $2\pi < |z| < 4\pi$.

(c). Define an *analytic* (holomorphic) function. Examine whether the following functions are analytic: (i) $\mathrm{Im}\, z$, (ii) $|z|^2$, (iii) $\ln|z|$.

[6 + 4 + 6]

(2) (a). Explain the notion of the *Cauchy Principal Value* of an improper integral, using the example of the following improper integral:

$$\mathcal{I} = \int_{-1}^{2} \frac{dx}{(x-1)}$$

(b). Let f have an isolated singularity at the point $z = z_0$. In other words, f is analytic in a punctured neighbourhood of the point $z = z_0$. Then show that the residue of $f'(z)$ at $z = z_0$ is *zero*.

(c). A geodesic on a given surface is, by definition, a curve, lying on that surface, such that the distance between two points on that curve is as small as possible. (Thus for instance, on the plane $\mathbb{R}^2$, a geodesic is a straight line). Obtain the equation for the geodesics on S^2 (the 2-sphere of unit radius) and hence show that these are given by the great circles. [3 + 4 + 5]

(3). (a) Let (x_0, y_0) and (x_1, y_1) be two given points in the plane. (Assume, $x_0, x_1, y_0, y_1 > 0$ and $x_1 > x_0$). Amongst all the curves joining these two given points, find the one which generates the surface of minimum area when rotated about the X-axis.

(b). Evaluate the integral:

$$\int_0^\infty \frac{\sin^3 x}{x^3}\, dx$$

[6 + 6]

(4). (a). Show that of all plane closed curves with a given fixed total length (perimeter), the circle is the one that encloses the largest area.

(b). Let Γ be a smooth, simply-connected closed contour, and let $f(z)$ be analytic inside and on Γ, with the exception of (may be) a finite number of points inside of Γ. Then show that

$$\oint_\Gamma \frac{dz}{2\pi i} \frac{f'(z)}{f(z)} = \#\text{ Zeros of } f \ - \ \#\text{ Poles of } f, \text{ (counted with multiplicity)}$$

[6 + 6]

(5). (a) What are the Cauchy-Riemann conditions? Find all $v(x, y)$: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, such that for $z = x + iy$, the function:
$f(z) = (x^3 - 3xy^2) + iv(x, y)$ is analytic.

(c). Compute the residue(s) of the following function at all its poles, including infinity.

$$f(z) = \left(\frac{z^{n-1}}{z^n - 1}\right)$$

[5 + 7]

(6) (a) Evaluate the integral (any ONE):

$$\int_0^\infty \frac{dx}{(1 + x^n)} \qquad OR \qquad \int_{-\infty}^\infty \frac{\cos x}{(x^2 + a^2)^2} dx \qquad (a > 0).$$

(b). Let $f(z) = \frac{z^5}{|z|^4}$ $(z \neq 0)$, and $f(0) = 0$. Show that the real and imaginary parts of f does satisfy Cauchy-Riemann equations at $z = 0$, yet f is *not* analytic at $z = 0$! Why?

(c). Consider the scenario of familiar Lagrangian dynamics as a direct application of the principles of the calculus of variations. Show that a change of the Lagrangian $L(q, \dot{q}, t)$ by a total derivative of some function $F(q, t)$:

$$L(q, \dot{q}, t) \longrightarrow \tilde{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF(q, t)}{dt}$$

does not alter the Euler-Lagrange equations of motion. (Prove this). What would happen if the function F was also to depend on $\dot{q}$? [3 + 4 + 5]